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On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields. V.

— Case of Interacting Electromagnetic and Meson Fields.—

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Introduction

According to the general program proposed in the first paper, Koba, Tati and one of the present authors have given in the second paper a perfectly relativistic formulation of the quantum electrodynamics, which deals with the electromagnetic field interacting with the electron field. In this paper we shall apply the same method to the meson field interacting with the electromagnetic field.

In doing this some generalizations of the treatment in I and II are necessary, because in I and II only such cases are considered in which the interaction energy of the fields under consideration is a scalar quantity whose densities at two different world points are commutable, provided that these points lie outside each other's light cones not only at a finite distance but also at an infinitesimal distance from each other. As stated in I, this is the case when the interaction part of the Lagrange function does not contain the time derivatives of the potentials describing the fields. However, in the cases of the meson fields, scalar as well as vectorial, interacting with electromagnetic field or the nucleon field this simplifying fact does not hold and the interaction energies are not a scalar. At the same time, the commutablility of the energy densities at two world points breaks down when these points lie adjacent to each other.

We shall show in this paper that, notwithstanding this complicating

(1) S. Tomonaga: Riken Ihô (Bul. IPCR.), 22 (1943) 545, Progr. Theor. Phys. 1 (1946), 27. This paper will be cited as I.

(2) Z. Koba, T. Tati and S. Tomonaga: Progr. Theor. Phys. 2 (1947), 101. This paper will be cited as II.

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situation, we can still build up a theory similar to that described in I and II. We first treat the simple case of the scalar or the pseudoscalar meson field interacting with the electromagnetic field, and then go over into the more complicated case of the vector or the pseudovector meson field.

A. Case of Scalar or Pseudoscalar Meson Field.

§ 1. Density of interaction energy. Four-dimensional commutation relations between field quantities.

Let L denote the Lagrange density of the total system. This consists of L_R and L_{SM}, L_R being the Lagrange density of the free electromagnetic field and L_{SM} that of the scalar or the pseudoscalar meson field including the terms representing the interaction with the electromagnetic field. Thus:

$$L = L_R + L_{SM}. \tag{1·1}$$

In this paper, it is more convenient to use the notations of the tensor calculus, not those of the vector calculus as we have done in II. Further we use the imaginary unit for the time coordinate:

$$x_4 = ict = ix_0 . \tag{1·2}$$

Then we have

$$\left.\begin{aligned} L_R &= -\frac{1}{8\pi}\left\{\frac{1}{2}F_{\mu\nu}F_{\mu\nu} + \left(\frac{\partial A_\mu}{\partial x_\mu}\right)^2\right\} \\ \text{with}\quad F_{\mu\nu} &= \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}, \qquad \mu, \nu = 1, 2, 3, 4 \end{aligned}\right\} \tag{1·3}$$

where A_i, $(i=1, 2, 3)$ denote the components of the vector potential, and $A_4 = iA_0$, A_0 being the scalar potential. We have further

$$L_{SM} = -\frac{1}{4\pi}\left\{\left(\frac{\partial \phi^*}{\partial x_\mu} + \frac{ie}{\hbar c}A_\mu \phi^*\right)\left(\frac{\partial \phi}{\partial x_\mu} - \frac{ie}{\hbar c}A_\mu \phi\right) + \varkappa^2 \phi^* \phi\right\}, \tag{1·4}$$

ϕ being the potential describing the scalar (or the pseudoscalar) meson field. The Greek suffixes σ, β, μ, ν etc. take four values 1, 2, 3 and 4, whereas the Latin suffixes run from 1 to 3 only.

If now canonically conjugate quantities of A_μ, ϕ and ϕ^* be denoted by

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A_μ^+, ϕ^+ and ϕ^{*+} respectively, we have

$$\left.\begin{aligned} A_\mu^+ &= \frac{\partial L_R}{\partial \dfrac{\partial A_\mu}{\partial x_4}} = -\frac{1}{4\pi}F_{4\mu} \\ \phi^+ &= \frac{\partial L_{SR}}{\partial \dfrac{\partial \phi}{\partial x_4}} = -\frac{1}{4\pi}\left(\frac{\partial \phi^*}{\partial x_4} + \frac{ie}{\hbar c}A_4\phi^*\right) \\ \phi^{*+} &= \frac{\partial L_{SM}}{\partial \dfrac{\partial \phi^*}{\partial x_4}} = -\frac{1}{4\pi}\left(\frac{\partial \phi}{\partial x_4} - \frac{ie}{\hbar c}A_4\phi\right). \end{aligned}\right\} \qquad (1\cdot5)$$

The canonical energy density of the system is obtained by the usual procedure:

$$\begin{aligned} H &= A_\mu^+\frac{\partial A_\mu}{\partial x_4} + \phi^+\frac{\partial \phi}{\partial x_4} + \frac{\partial \phi^*}{\partial x_4}\phi^{*+} - L \\ &= H_R + H_{SM}. \end{aligned} \qquad (1\cdot6)$$

In $(1\cdot6)$ the first term H_R is the energy density for the electromagnetic field alone, and the second term H_{SM} is that of the meson field including the interaction terms with the eleetromagnetic field. As in the following we make no use of H_R, we give here only the explicit expression for H_{SM}. It is defined by

$$H_{SM} = \phi^+\frac{\partial \phi}{\partial x_4} + \frac{\partial \phi^*}{\partial x_4}\phi^{*+} - L_{SM}. \qquad (1\cdot7)$$

This H_{SM} can be separated into two parts:

$$H_{SM} = \mathring{H}_{SM} + H_{\text{int}} \qquad (1\cdot8)$$

with

$$\mathring{H}_{SM} = \frac{1}{4\pi}\left\{\frac{\partial \phi^*}{\partial x_i}\frac{\partial \phi}{\partial x_i} - (4\pi)^2\phi^+\phi^{*+} + \varkappa^2\phi^*\phi\right\} \qquad (1\cdot9)$$

and

$$\begin{aligned} H_{\text{int}} = \frac{1}{4\pi}\Bigg[\left(\frac{ie}{\hbar c}\right)\Bigg\{A_j\left(\phi^*\frac{\partial \phi}{\partial x_i} - \frac{\partial \phi^*}{\partial x_i}\phi\right) \\ -4\pi A_4(\phi^*\phi^{*+} - \phi^+\phi)\Bigg\} - \left(\frac{ie}{\hbar c}\right)A_i\phi^*\phi\Bigg]. \end{aligned} \qquad (1\cdot10)$$

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The first term $\overset{\circ}{H}_{SM}$ in (1·8) is independent of the electromagnetic potentials and represents the energy density of the free meson field, while the second term H_{int} contains the mesonic as well as the electromagnetic potentials and represents density of the interaction energy between these two fields. The expression (1·9) and (1·10) are obtained by the usual procedure by substituting (1·5) into (1·7).

According to our general schem developed in I, we now transform the field quantities by means of the unitary operator

$$U=e^{(i/\hbar)(\overline{H}_R+\overline{\overset{\circ}{H}}_{SM})t} \tag{1·11}$$

Then the transformed quantities satisfy the fleld equations for the free fields. Especially ϕ^{*+} and ϕ^{+} in H_{int} can be replaced by $-\frac{1}{4\pi}\frac{\partial\phi}{\partial x_4}$ and $-\frac{1}{4\pi}\frac{\partial\phi^*}{\partial x_4}$ respectively. Thus we have

$$H_{\text{int}}=\frac{1}{4\pi}\left\{\left(\frac{ie}{\hbar c}\right)A_\alpha\left(\phi^*\frac{\partial\phi}{\partial x_\alpha}-\frac{\partial\phi^*}{\partial x_\alpha}\phi\right)-\left(\frac{ie}{\hbar c}\right)^2 A_i^2\phi^*\phi\right\}. \tag{1·12}$$

Since, in this way, the field quantities satisfy the field equations for the free fields, we can obtain the four-dimentional commutation relations between them:

$$\left.\begin{aligned}&[A_\alpha(X),\ A_\beta(X')]=-4\pi i\hbar c\ \delta_{\alpha\beta}D_{\text{I}}(X-X')\\&[\phi(X),\ \phi^*(X')=-4\pi i\hbar c\ D_{\text{II}}(X-X')\\&[\phi(X),\ \phi(X')]=[\phi^*(X),\ \phi^*(X')]=0,\end{aligned}\right\} \tag{I}$$

where D_{I} and D_{II} are the so-called four-dimensional delta-functions belonging respectively to the electromagnetic field and the mesonic field.

Observing (1·12), we find that the interaction energy density is not a scalar. Although the first term in (1·12) is a scalar, the second term is not, since in the summation $A_j^2=A_1^2+A_2^2+A_3^2$ the fourth component of the potential is abscent. This situation prevent us from applying directly the method developed in I. As we shall show in the next paragraph we have beside this a further difficulty: our H_{int} does not satisfy $[H_{\text{int}}(X),\ H_{\text{int}}(X')]=0$, which would be necessary for the integrability of the generalized Schrödinger equation.

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§ 2. Commutation relation between energy densities at two different worldpoints.

We calculate in this paragraph the commutator $[H_{\rm int}(X),\ H_{\rm int}(X')]$. For simplicity, we denote $H_{\rm int}(X)$ by H, $H_{\rm int}(X')$ by H', $A(X)$ by A, $A(X')$ by A' etc. Then we have

$$
\begin{aligned}
[H,\ H'] = {} & \frac{1}{16\pi^2}\left(\frac{ie}{\hbar c}\right)^2\Big[A_\alpha\Big(\phi^*\frac{\partial\phi}{\partial x_\alpha}-\frac{\partial\phi^*}{\partial x_\alpha}\phi\Big) \\
& -\left(\frac{ie}{\hbar c}\right)A_j^2\phi^*\phi,\ A'_\lambda\Big(\phi^{*\prime}\frac{\partial\phi'}{\partial x'_\lambda}-\frac{\partial\phi^{*\prime}}{\partial x_\lambda{}'}\phi'\Big)-\left(\frac{ie}{\hbar c}\right)A_k{}'\phi^{*\prime}\phi'\Big] \\
= {} & \frac{1}{16\pi^2}\left(\frac{ie}{\hbar c}\right)^2\Big\{\Big[A_\alpha\Big(\phi^*\frac{\partial\phi}{\partial x_\alpha}-\frac{\partial\phi^*}{\partial x_\alpha}\phi\Big),\ A'_\lambda\Big(\phi^{*\prime}\frac{\partial\phi'}{\partial x_\lambda{}'}-\frac{\partial\phi^{*\prime}}{\partial x_\lambda{}'}\phi'\Big)\Big] \\
& -\left(\frac{ie}{\hbar c}\right)\Big[A_\alpha\Big(\phi^*\frac{\partial\phi}{\partial x_\alpha}-\frac{\partial\phi^*}{\partial x_\alpha}\phi\Big),\ A_k{}'\phi^{*\prime}\phi'\Big] \\
& -\left(\frac{ie}{\hbar c}\right)\Big[A_j^2\phi^*\phi,\ A_\lambda{}'\Big(\phi^{*\prime}\frac{\partial\phi'}{\partial x'_\lambda}-\frac{\partial\phi^{*\prime}}{\partial x_\lambda{}'}\phi'\Big)\Big] \\
& +\left(\frac{ie}{\hbar c}\right)^2[A_j^2\phi^*\phi,\ A_k'^2\phi^*\phi']\Big\}. \qquad (2\cdot1)
\end{aligned}
$$

We now calculate each commutator on the right-hand side of $(2\cdot1)$. Then, we see first that the fourth term on the right-hand side contains $D(X-X')$ but not its derivatives. As we are interested only in such pairs of points one of which is space-like with respect to the other, such terms containing no derivatives of $D(X-X')$ can be replaced by 0. Next we see that the second and the third terms give rise to the term of the form

$$
-4\pi e\{A_\alpha A_k'^2(\phi^*\phi'-\phi^{*\prime}\phi)-A_\alpha{}'A_k^2(\phi^{*\prime}\phi-\phi^*\phi')\}\,\frac{\partial D_{\rm II}(X-X')}{\partial x_\alpha},
$$

which vanishes, because $\dfrac{\partial D_{\rm II}(X-X')}{\partial x_\alpha}$ has non-vanishing value only when $X=X'$ (provided that X and X' are a space-like pair) where the factor $\{\ldots\ldots\}$ vanishes.

In this way only the first term on the right-hand side of $(2\cdot1)$ gives a non-vanishing contribution:

$$
[H,\ H'] = -\frac{ie^2}{4\pi\hbar c}\Big\{A_\alpha A'_\beta\phi^*\phi'\frac{\partial^2 D_{\rm II}(X-X')}{\partial x_\alpha\partial x_\beta{}'}-A'_\alpha A_\beta\phi^{*\prime}\phi\frac{\partial^2 D_{\rm II}(X'-X)}{\partial x_\alpha{}'\partial x_\beta}\Big\}. \quad (2\cdot2)
$$

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A though the right-hand side of (2·2) vanishes for the pair of points X and X' when they lie at a finite distance from each other and outside each-other's light cones, it has a non-vanishing value when they are adjacent to each other. Namely, the function $\frac{\partial^2 D_{\text{II}}(X-X')}{\partial x_\alpha \partial x_\beta'} = -\frac{\partial^2 D_{\text{II}}(X'-X)}{\partial x_\alpha' \partial x_\beta}$ has a non-vanishing value when X' is adjacent to X.

Since $\frac{\partial^2 D_{\text{II}}(X-X')}{\partial x_\alpha \partial x_\beta'}$ has thus a non-vanishing value only when X and X' are adjacent to each other, (2·2) can be also expressed in the form:

$$[H, H'] = -\frac{ie^2}{4\pi\hbar c}\left\{A_\alpha A_\beta \phi^* \phi \frac{\partial^2 D_{\text{II}}(X-X')}{\partial x_\alpha \partial x_\beta'} - A_\alpha' A_\beta' \phi^{*\prime} \phi' \frac{\partial^2 D_{\text{II}}(X'-X)}{\partial x_\alpha' \partial x_\beta}\right\}. \quad (2\cdot3)$$

This form is more convenient than (2·2) for our latter purpose.

§ 3. Derivation of the generalized Schrodinger equation.

As stated in the preceding paragraph, our energy density H does not satisfy $[H, H']=0$. But this does in no way prevent that the Ψ–vector, transformed by means of the unitary operator U in (1·11), satisfies the

$$\left\{\bar{H}_{\text{int}} + \frac{\hbar}{i}\frac{\partial}{\partial t}\right\}\Psi = 0. \quad (3\cdot1)$$

The situation is only so far complicated that we can not go over from (3·1) immediately into

$$\left\{H(P) + \frac{\hbar}{i}\frac{\delta}{\delta C_P}\right\}\Psi[C] = 0, \quad (3\cdot2)$$

because this necessitates the condition $[H(P), H(P')]=0$.

In such a situation we proceed in the following manner: we introduce the operator $A_X[C]$, a function of world point X in one hand* and a func-

* Similarly as in II, X, X', X'', ...denote arbitrary world points and P, P', P'', ... such world points lying on C. We use for both kinds of points the same letters x_μ, x'_μ, x''_μ, ...to denote their coordinates, but this makes no confusion in the following consideration.

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tional of the variable surface C on the other, such that it vanishes, in the first place, when C is reduced into a plane parallel to the xyz-plane, and it satisfies, in the second place,

$$\left[H(P)+A_P[C]+\frac{\hbar}{i}\frac{\delta}{\delta C_P},\ H(P')+A_{P'}[C]+\frac{\hbar}{i}\frac{\delta}{\delta C_{P'}}\right]=0. \quad (3\cdot3)$$

Then, the functional differential equation

$$\left\{H(P)+A_P[C]+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right\}\Psi[C]=0 \quad (3\cdot4)$$

is integrable, and contains the equation (3·1) as the special case in which the surface C is reduced into the plane C_t parallel to the xyz-plane and intersecting the time axis at t. Further, if it is possible to obtain $A_X[C]$ such that the expression $H(P)+A_P[C]$ becomes a scalar, then our equation (3·4) is relativistically invariant and has a definite meaning without referring to any Lorentz frame of reference.

We shall show that such a choice of $A_X[C]$ is in fact possible.

The relation (3·3) is first expressed in the form:

$$\frac{\hbar}{i}\left\{\frac{\delta A_P[C]}{\delta C_{P'}}-\frac{\delta A_{P'}[C]}{\delta C_P}\right\}+[H(P'),\ H(P)]$$
$$+[A_{P'}[C],A_P[C]]+[A_{P'}[C],\ H(P)]+[H(P'),\ A_P[C]]=0. \quad (3\cdot5)$$

Now, this equations is satisfied if we can fined $A_P[C]$ in such a way that its " rotation " satisfies

$$\frac{\delta A_P[C]}{\delta C_{P'}}-\frac{\delta A_{P'}[C]}{\delta C_P}=\frac{i}{\hbar}[H(P),\ H(P')]$$
$$=-\frac{c}{4\pi}\left(\frac{ie}{\hbar c}\right)^2\left\{A_\alpha A_\beta\phi^*\phi\frac{\partial^2 D_{\rm II}(P-P')}{\partial x_\alpha\partial x_\beta'}-A_\alpha' A_\beta'\phi^{*\prime}\phi'\frac{\partial^2 D_{\rm II}(P'-P)}{\partial x_\alpha'\partial x_\beta}\right\}, \quad (3\cdot6)$$

and further that the commutation relations

$$\left.\begin{array}{l}[A_{P'}[C],\ A_P[C]]=0\\ [A_{P'}[C],\ H(P)]=0\end{array}\right\} \quad (3\cdot7)$$

are fulfilled.

The " rotation " equation (3·6) can be satisfied when we can find the solution of

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$$\frac{\delta A_P[C]}{\delta C_{P'}} = -\frac{c}{4\pi}\left(\frac{ie}{\hbar c}\right)^2 A_\alpha A_\beta \phi^* \phi \frac{\partial^2 D_{11}(P-P')}{\partial x_\alpha \partial x_\beta'}, \quad (3\cdot6')$$

but this equation can be solved immediately by applying the theorem given in II. (i.e. the formula (3·7)) We get, namely,

$$A_P[C] = \frac{1}{4\pi}\left(\frac{ie}{\hbar c}\right)^2 \int_C A_\alpha A_\beta \phi^* \phi N_\beta' \frac{\partial D_{11}(P-P')}{\partial x_\alpha} dF_{P'} + K \quad (3\cdot8)$$

where N'_β denotes the component of the unit vector normal to the surface C at P and pointing to the future. The integration constant K is an arbitrary operator independent of C.

We can carry out the integration on the right-hand side of (3·8). This is done in the reference system whose space axes are tangent to C at P. Denotlng the components of vectors in this system by barred suffixes, and noting that in this system $\dfrac{\partial D_{11}(P-P')}{\partial x_\alpha}$ has the non-vanishing component $-i\delta(\vec{x}-\vec{x}')$ only in the 4–th direction, we obtain

$$\int_C A_\alpha A_\beta \phi^* \phi N'_\beta \frac{\partial D_{11}(P-P')}{\partial x_\alpha} dF_{P'} = \int_C A_{\bar{4}} A_{\bar{\beta}} \phi^* \phi N_{\bar{\beta}} (-i) \delta(\vec{x}-\vec{x}') dF_{P'},$$

or, denoting $N_{\bar{\alpha}}$ the component of the unit vector normal to C at P and pointing to the future, this can be written in the form

$$= -\int_C (A_{\bar{\alpha}} N_{\bar{\alpha}})(A_{\bar{\beta}} N_{\bar{\beta}}') \phi^* \phi \delta(\vec{x}-\vec{x}') dF_{P'},$$

because $N_{\bar{\alpha}}$ has values (0, 0, 0, i). On carrying out the integration this gives

$$= -(A_{\bar{\alpha}} N_{\bar{\alpha}})(A_{\bar{\beta}} N_{\bar{\beta}}) \phi^* \phi,$$

which, returning to the general coordinate system, gives rise to

$$= -(A_\alpha N_\alpha)^2 \phi^* \phi.$$

We obtain in this way

$$A_P[C] = -\frac{1}{4\pi}\left(\frac{ie}{\hbar c}\right)^2 (N_\alpha A_\alpha)^2 \phi^* \phi + K.$$

The integration constant K is now so chosen that $A_P[C]$ vanishes when C is reduced into a plane parallel to the xyz–plane. Thus we have to put

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$$K=-\frac{1}{4\pi}\left(\frac{ie}{\hbar c}\right)^2 A_4^2\phi^*\phi,$$

so that we obtain the required solution of (3·6'):

$$A_P[C]=-\frac{1}{4\pi}\left(\frac{ie}{\hbar c}\right)^2\{(N_\alpha A_\alpha)^2\phi^*\phi+A_4^2\phi^*\phi\}. \qquad (3\cdot 9)$$

Now, we have to prove that this $A_P[C]$ satisfies (3·7). The first relation of (3·7) is satisfied, because the commutator on the right-hand side contains only $D(P-P')$ but no derivatives of D. This is obvious from the fact that (3·9) contains no derivatives of the field quantities. The fact that the second relation in (3·7) is also satisfied can be shown by the direct calculation:

$$[A_{P'}[C], H(P)]=\text{const. } A_\beta\{N_\alpha' A_\alpha')^2(\phi\ \phi'-\phi^{*\prime}\phi) + A_4'(\phi^*\phi'-\phi^{*\prime}\phi)\}\frac{\partial D_{\text{II}}(P-P')}{\partial x_\beta},$$

the right-hand side of this expression vanishes, because $\frac{\partial D_{\text{II}}(P-P')}{\partial x_\beta}$ has non-vanishing value only when $X=X'$, where the faetor $\{\ldots\ldots\}$ vanishes.

In this way we can find the quantity $A_P[C]$ to be added to $H(P)$. Denoting this sum by $H_P[C]$:

$$H_P[C]=H(P)+A_P[C] \qquad (3\cdot 10)$$

we have

$$H_P[C]=\frac{1}{4\pi}\left[\left(\frac{ie}{\hbar c}\right)A_\alpha\left(\phi^*\frac{\partial\phi}{\partial x_\alpha}-\frac{\partial\phi^*}{\partial x_\alpha}\phi\right) -\left(\frac{ie}{\hbar c}\right)^2\phi^*\phi\{A_\alpha^2+(N_\alpha A_\alpha)^2\}\right]. \qquad (3\cdot 11)$$

It is to be noted that the non-scalar term $A_j^2\phi^*\phi$ in $H(P)$ and the non-scalar term $A_4^2\phi^*\phi$ in $A_P[C]$ just make up the scalar term $A_\alpha^2\phi^*\phi$, thus giving the scalar quantity $H_P[C]$.

The obtained scalar quantity $H_P[C]$ satisfies now the condition

$$\left[H_P[C]+\frac{\hbar}{i}\frac{\delta}{\delta C_P},\ H_{P'}[C]+\frac{\hbar}{i}\frac{\delta}{\delta C_{P'}}\right]=0 \qquad (3\cdot 12)$$

which guarantees the integrability of the equation

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$$\left.\begin{aligned}&\left\{H_P[C]+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right\}\Psi[C]=0\\ \text{with}\quad &H_P[C]=\frac{1}{4\pi}\Big[\Big(\frac{ie}{\hbar c}\Big)A_\alpha\Big(\phi^*\frac{\partial\phi}{\partial x_\alpha}-\frac{\partial\phi^*}{\partial x_\alpha}\phi\Big)\\ &\qquad-\Big(\frac{ie}{\hbar c}\Big)^2\phi^*\phi\{A_\alpha^2+(N_\alpha A_\alpha)^2\}\Big]\end{aligned}\right\}\qquad\text{(II)}$$

According to our construction of $H_P[C]$, this equation is reduced into the ordinary Schrödinger equation (3·1) when the surface C is reduces into the plane C_t parallel to the xyz-plane and intersecting the time axis at t. Further, the equation (II) has a definite meaning without referring to any Lorentz frame of reference, so that this equation can be regarded as the required generalization of the Schrödinger equation.

§ 4. Auxiliary condition.

As we have seen in II, the auxiliary condition in the case of the quantum electrodynamics has the form

$$\left\{\frac{\partial A_\alpha}{\partial x_\alpha}+\int_C J'_\alpha N'_\alpha D_1(P'-X)dF_{P'}\right\}\Psi[C]=0 \qquad (4\cdot1)$$

where $J_\alpha(X)$ is the four-current density at the world point X. In our case of the meson field the auxiliary condition has also the form of (4·1) with the current density

$$J_\alpha(X)=\frac{ie}{\hbar c}\Big(\frac{\partial\phi^*}{\partial x_\alpha}\phi-\phi^*\frac{\partial\phi}{\partial x_\alpha}\Big). \qquad (4\cdot2)$$

Notice that here the current density for the free meson field is to be used. The suggested auxiliary condition for the scalar meson field will be thus

$$\left.\begin{aligned}&\Xi_X[C]\Psi[C]=0\\ \text{with}\quad &\Xi_X[C]=\frac{\partial A_\alpha}{\partial x_\alpha}+\frac{ie}{\hbar c}\int_C\Big(\frac{\partial\phi^{*\prime}}{\partial x_\alpha'}\phi'-\phi^{*\prime}\frac{\partial\phi'}{\partial x_\alpha'}\Big)N_\alpha'D_1(P'-X)dF_{P'}.\end{aligned}\right\}\ \text{(III)}$$

In order that (III) can really be used as the auxiliary condition, we must prove its compatibility:

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$$\left.\begin{array}{l}\left[H_P[C]+\dfrac{\hbar}{i}\dfrac{\delta}{\delta C_P},\quad \Xi_{X'}[C]\right]=0\\ {[\Xi_X[C],\quad \Xi_{X'}[C]]=0.}\end{array}\right\} \tag{4·3}$$

In order to verify the first relation of (4·3), we notice first

$$\left[H_P[C],\ \frac{\partial A_\mu'}{\partial x_\mu'}\right]=e\left(\frac{\partial\phi^*}{\partial x_\alpha}\phi-\phi^*\frac{\partial\phi}{\partial x_\alpha}\right)\frac{\partial D_{\mathrm{I}}(P-X')}{\partial x_\alpha}$$
$$+\frac{2ie^2}{\hbar c}\{A_\alpha+(A_\alpha N_\beta)N_\alpha\}\phi^*\phi\frac{\partial D_{\mathrm{I}}(P-X')}{\partial x_\alpha} \tag{4·4}$$

Next, using the formula

$$\left[\frac{\partial\phi^*}{\partial x_\alpha}\phi-\phi^*\frac{\partial\phi}{\partial x_\alpha},\ \frac{\partial\phi^{*\prime\prime}}{\partial x_\mu''}\phi''-\phi^{*\prime\prime}\frac{\partial\phi''}{\partial x_\mu''}\right]$$
$$=4\pi i\hbar c\left\{\left(\phi^{*\prime\prime}\frac{\partial\phi}{\partial x_\alpha}-\frac{\partial\phi^*}{\partial x_\alpha}\phi''\right)\frac{\partial D_{\mathrm{II}}(X'-X)}{\partial x_\mu''}\right.$$
$$\left.-\left(\phi\frac{\partial\phi''}{\partial x_\mu''}+\frac{\partial\phi^{*\prime\prime}}{\partial x_\mu''}\phi\right)\frac{\partial D_{\mathrm{II}}(X''-X)}{\partial x_\alpha''}-(\phi^*\phi+\phi^{*\prime\prime}\phi)\frac{\partial^2 D_{\mathrm{II}}(X''-X)}{\partial x_\beta''\partial x_\mu''}\right\} \tag{4·5}$$

and

$$\left[\phi^*\phi,\ \frac{\partial\phi^{*\prime\prime}}{\partial x_\mu''}\phi''-\phi^{*\prime\prime}\frac{\partial\phi''}{\partial x_\mu''}\right]=0,$$

we calculate

$$\left[H_P[C],\ \left(\frac{ie}{\hbar c}\right)\int_C\left(\frac{\partial\phi^{*\prime\prime}}{\partial x_\mu''}\phi''-\phi^{*\prime\prime}\frac{\partial\phi''}{\partial x_\mu''}\right)N_\mu''D_{\mathrm{I}}(P''-X)dF_{P''}\right]$$

Carrying out the integration over P'' in the reference system whose space axes are tangent to C at P'', we find then

$$\left[H_P[C],\ \left(\frac{ie}{\hbar c}\right)\int_C\left(\frac{\partial\phi^{*\prime\prime}}{\partial x_\mu''}\phi''-\phi^{*\prime\prime}\frac{\partial\phi''}{\partial x_\mu''}\right)N_\mu''D_{\mathrm{I}}(P''-X')\,dF_{P''}\right]$$
$$=-\frac{2ie}{\hbar c}\{A_\alpha+(A_\beta N_\beta)N_\alpha\}\frac{\partial D_{\mathrm{I}}(P-X')}{\partial x_\alpha}. \tag{4·6}$$

Thus, the results (4·4) and (4·6) give rise to

$$[H_P[C],\ \Xi_{X'}[C]]=e\left(\frac{\partial\phi^*}{\partial x_\alpha}\phi-\phi^*\frac{\partial\phi}{\partial x_\alpha}\right)\frac{\partial D_{\mathrm{I}}(P-X')}{\partial x_\alpha}. \tag{4·7}$$

Now, applying the formula obtained in II:

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$$\frac{\delta}{\delta C_P}\int_c G'_\mu N'_\mu dF_{P'} = -c\frac{\partial G_\mu}{\partial x_\mu}$$

and noting the equation of continuity for the current:

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial \phi^*}{\partial x_\mu}\phi - \phi^*\frac{\partial \phi}{\partial x_\mu}\right)=0,$$

we obtain

$$\left[\frac{\hbar}{i}\frac{\delta}{\delta C_P},\ \Xi_{X'}[C]\right] = -e\left(\frac{\partial \phi^*}{\partial x_\mu}\phi - \phi^*\frac{\partial \phi}{\partial x_\mu}\right)\frac{\partial D_1(P-X')}{\partial x_\mu}. \quad (4\cdot8)$$

Summing up the results (4·7) and (4·8), we obtain the required relation:

$$\left[H_P[C] + \frac{\hbar}{i}\frac{\delta}{\delta C_P},\ \Xi_{X'}[C]\right]=0.$$

We must next prove the second relation of (4·3). We have first

$$\left[\frac{\partial A_\alpha}{\partial x_\alpha},\ \frac{\partial A_\lambda'}{\partial x_\lambda'}\right] = -4\pi i\hbar c\,\delta_{\alpha\lambda}\frac{\partial^2 D_1(X-X')}{\partial x_\alpha \partial x_\lambda'}$$
$$=4\pi i\hbar c\frac{\partial^2 D_1(X-X')}{\partial x_\alpha^2},$$

but the right-hand side of this equation vanishes because $D_1(X)$ satisfies $\Box D_1=0$. Next, by applying (4·5) we see immediately that

$$\left[\int_c\left(\phi^{*''}\frac{\partial \phi''}{\partial x_\alpha''} - \frac{\partial \phi^{*''}}{\partial x_\alpha''}\phi''\right)N_\alpha'' D_1(P''-X)dF_{P''},\right.$$
$$\left.\int_c\left(\phi^{*'''}\frac{\partial \phi'''}{\partial x_\lambda'''} - \frac{\partial \phi^{*'''}}{\partial x_\lambda'''}\phi'''\right)N_\lambda''' D_1(P'''-X')dF_{P'''}\right]=0.$$

These two relations show that the required relation

$$[\Xi_X[C],\ \Xi_{X'}[C]]=0.$$

is really satisfied.

Having thus proved that our auxiliary condition is compatible, our next task is to show that the condition (III) gives in fact the correct Maxwell equation when we go over into the ordinary formulation. This problem, together with the problem of eliminating the auxiliary condition, will

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be given in another place.*

In this way we have shown that the perfectly relativistic formulation is possible also in the case of scalar (or pseudoscalar) meson field interacting with the electromagnetic field despite the difficulties mentioned in the foregoing paragraphs. We can get over these difficulties by adding to the interaction energy density $H(P)$ a new term whose "rotation" cancels the commutator of the energy density. It is a remarkable fact, that in this way not only one obtains the quantity satisfying the condition of integrability, but also non-scalar density $H(P)$ can be supplemented into the scalar quantity. Thus the obtained quantity $H_P[C]$ can be used as the characterizing operator of the system to be used in the generalized Schrödinger equation. This characteristic operator is not a point function of the space-time but contains the direction cosines of the normal to the surface C, so that it is rather a function of the variable surface-element in the space-time world.

This last fact introduces a further restriction for the shape of $d\omega$ used in constructing the g.t.f. $T[C_2, C_1]=\prod_{C_1}^{C_2}\left\{1-\frac{\hbar}{i}H_P[C]d\omega\right\}$ over the restriction introduced in I. Namely, it is required that, not only $d\omega$ should be surrounded by two space-like surfaces, but the shape of this surfaces should be such that the difference between directions of normals drawn at any two points on these surfaces is infinitesimal of the same order as the volume $d\omega$. Thus our elementary regions must be of scale form flat in time-like directions, so that, according to our theory, the world has, so to speak, a laminar and not a granular structure.

(to be continued)

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