CONTEMPORARY MATHEMATICS

315

Quantization, Poisson Brackets and Beyond

London Mathematical Society Regional Meeting and Workshop on Quantization, Deformations, and New Homological and Categorical Methods in Mathematical Physics
July 6–13, 2001
University of Manchester Institute of Science and Technology, Manchester, United Kingdom

Theodore Voronov
Editor



American Mathematical Society

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Preface

A characteristic feature of current developments in mathematics — and definitely a trend for the future — is the emergence of various "non-classical" geometric and algebraic structures. It is impossible to list them all: from one direction come "spaces" and "groups" that are not spaces nor groups in the usual sense but are rather super or quantum objects; from another direction come multiple categories (with various levels of strictness) and multiple groupoids; thirdly, there come "homotopy relatives" of Lie and associative algebras; and there are many other examples. Some of these structures actually came into being decades ago, but have only now started to receive due appreciation (supermanifolds and supergroups; Lie groupoids and algebroids); others have gotten a new impetus due to new powerful examples (Hopf algebras from quantum groups), or have undergone an amazing rebirth (operads and various homotopy structures). Classical objects like Lie groups and Lie algebras of course remain important, but they are no longer felt sufficient. The motivation for new structures comes, to a great extent, from mathematical and theoretical physics. Experts debate whether some of the "new physics" could actually be related to physical reality, but no one can doubt that it has supplied a plentitude of new mathematical problems, results, methods and inspirations. Apart from that, it seems completely certain to the writer that the new geometric and algebraic structures are not merely a product of fashion. They possess a vitality of their own, and, as is becoming more and more evident, they have natural motivations from and firm links with the "classical" differential geometry and algebraic topology of the twentieth century.

Quantization theory and related mathematical areas (like deformation theory) are in the focus of rapid development. As landmarks one can see the works by Boris Fedosov on the quantization of general symplectic manifolds and by Maxim Kontsevich on the quantization of arbitrary Poisson manifolds. The latter work manifested the rebirth (or rethinking) of the deformation theory of various mathematical structures (cf. "Nijenhuis's program" of the 1960s) and attracted general attention to such "non-classical" tools as odd brackets, homotopy algebras, operads, etc., which had already been in use in the mathematical physics of recent years. These results gave an enormous boost to the field, which is now extensively developing. Independently, a "cohomological physics", to use Jim Stasheff's word, grew out of gauge-invariant field theory. An example is the renowned Batalin–Vilkovisky quantization procedure. Since about 1990, there have also been rapid developments in the theory of Lie algebroids, Lie bialgebroids and Poisson–Lie groupoids. These theories often provide a semi-classical form of quantization, which mediates between classical mathematics and fully quantum theories.

The works collected in this volume are based on the talks at the Manchester Meeting of the London Mathematical Society, 6 July 2001, followed by an LMS-sponsored international workshop, 7–13 July 2001, the topic and title of which was *Quantization, deformations, and new homological and categorical methods in mathematical physics.* Some of the papers are very close accounts of the actual talks; others present new results developing those communicated at the meetings.

The papers are centered on the following topics: quantization in various aspects, Poisson brackets and structures "beyond", — symplectic supermanifolds, operads, Lie groupoids and Lie (bi)algebroids, algebras with n-ary operations, — to name a few. I have tried to group the papers logically, at least as far as is practical. The style of exposition varies. Some papers are mainly addressed to experts, while others can also serve as an accessible introduction and as reference texts. Hopefully, this volume will be useful to a broad reading audience of researchers in geometry, topology, algebra and mathematical physics.

ACKNOWLEDGEMENTS. There are several people and organizations that I would like to thank. In 2001 the London Mathematical Society started its regional meetings, of which the Manchester Meeting was one of the first. I thank the Society for the help in organization and for the financial support of the Meeting and workshop. The American Mathematical Society suggested to publish the proceedings, to which the LMS kindly gave its blessing. I want to thank Sergei Gelfand for the idea of this publication. Mike Prest and myself were the co-organizers of the meeting and workshop, Mike often taking the harder part of the task. I am very grateful to him. Thanks are due to UMIST (the University of Manchester Institute of Science and Technology) for providing conference facilities and various help. The greatest thanks are due to all the participants and especially to those who contributed to these proceedings. Many friends and colleagues helped me during the work on the book. I want to thank Alan Weinstein, Dmitry Gurevich, Alexander Karabegov, Dmitry Roytenberg and Jens Hoppe. Special thanks go to Kirill Mackenzie and John Rawnsley for crucial help, particularly at the final stages of the work.

Theodore Voronov
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Deformation quantization: pro and contra

Boris Fedosov

ABSTRACT. We list some advantages and drawbacks of deformation quantization.

Advantages:

(1) Deformation quantization gives a flexible quantization procedure which works for any classical phase manifolds: Kähler, symplectic or Poisson.
(2) The Hamiltonian reduction in the framework of deformation quantization goes almost similarly to the classical case.
(3) Admits generalizations to non-trivial coefficient bundles and for some infinite-dimensional manifolds.
(4) There is a trace functional giving rise to index theorems.

Drawbacks and questions:

(1) Difficulties with calculations.
(2) Is it possible to get predictions with the help of deformation quantization?
(3) Is there a reasonable substitution for spectral theory? For the Schrödinger evolution?

Introduction

In the last 5 years deformation quantization became an intensively developing field, especially after the seminal work of M. Kontsevich where he not only gave an answer to the longstanding question of deformation quantization, namely, the existence and classification of star-products for every Poisson manifold, but also discovered new directions and generalizations connected with the formality theorem. We all know that this work has found a wide resonance: there appeared a great number of publications caused by Kontsevich's paper, from a simple re-exposition of his results to completely new ideas leading to new proofs. There appeared also new names, let me mention A. Kattaneo, G. Felder, D. Tamarkin. I vividly imagine a picture: a powerful tractor plowing a field and a crowd of birds behind looking for warms. The Fields Medal awarded to Kontsevich in 1998 promoted deformation quantization to a fashionable field, so to say "science a la mode". Within the stream of information I have felt an increasing desire to analyze and valuate the deformation quantization as a whole: what is its place among different quantum theories, what are its merits and drawbacks, can we expect further intensive development and

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so on. I am deeply indebted to the organizers of this meeting who gave me the opportunity to speak here on this subject.

One can observe the ideas of deformation quantization at the very beginning of quantum mechanics. I mean a non-commutative algebra of quantum observables satisfying the correspondence principle and in the limit giving the classical commutative algebra of observables. The only thing which was unrealized for a rather long time is how to understand the limiting procedure to the classical mechanics, and, in particular, what is the precise meaning of the correspondence principle. And comparatively recently, in the late 70-s F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer suggested to consider quantum mechanics as a deformation theory. This point of view gave the maximal freedom for mathematicians with their rigor that often seems to tie hands. It may sound funny, but for physicists the idea of formal power series in h at first seemed a little bit strange. I remember I gave a lecture course on deformation quantization for graduate students of Moscow Institute of Physics and Technology, and one of the first questions was: "Don't you find that the formal power series look very abstract, it is difficult to operate with this notion". It took me some time to explain to them that this notion should be the most familiar for physicists because they simply compute the coefficients of various expansions and don't care of convergence or asymptotic nature of these expansions. In other words, they actually work with formal power series. This reminds me of one of Molière's characters who did not know that he spoke prose.

Unfortunately, untying our hands we loose important phenomena, since deformation quantization is unable to handle rapidly oscillating or rapidly decreasing exponents . Thus, the typical quantum effects, such as diffraction pictures or tunnelling, remain outside the scope of deformation quantization. This is the main contradiction which we come across again and again trying to realize the place of deformation quantization. On one hand, the maximal freedom in the algebra of quantum observables, on the other hand, a certainly incomplete description of quantum phenomena. In my talk I will try to explain this contradiction in more detail, illustrating it on several examples.

1. Basic Notions

Let me recall the basic notions of deformation quantization. We consider a Poisson manifold with an antisymmetric Poisson tensor α^{ij} which makes the commutative algebra of functions $C^\infty(M)$ into a Lie algebra with the bracket

$$\{u, v\} = \alpha^{ij}\partial_i u \partial_j v$$

satisfying the Jacobi identity and compatible with the commutative product

$$\{u, vw\} = \{u, v\}w + v\{u, w\}.$$

A particular case is a symplectic manifold, when the Poisson tensor is non-degenerate. Denoting ω_{ij} the inverse matrix to α^{ij}, one can see that the Jacobi identity is equivalent to the closedness of the form

$$\omega = \frac{1}{2}\omega_{ij}dx^i \wedge dx^j.$$

DEFINITION 1.1. A *deformation quantization* is an associative algebra structure on the space $A(M) := C^\infty(M)[[h]]$ of formal power series in a formal parameter h with respect to a product $*$ (called a *star-product*) with the properties:

(1)
$$a * b = \sum_{k=0}^{\infty} h^k C_k(a, b)$$
where C_k are bidifferential operators (the so-called locality property),

(2) the star-product is a formal deformation of the classical commutative product, that is
$$C_0(a, b) = ab,$$

(3)
$$C_1(a, b) - C_1(b, a) = ih\{a, b\},$$
(the so-called correspondence principle),

(4) (optional)
$$1 * a = a * 1 = a$$
for any $a \in A$

An *equivalence* of two star-products $*$ and $*'$ is an isomorphism between the algebras $(A, *)$ and $(A, *')$ given by a formal differential operator
$$Ta = a + hT_1 a + h^2 T_2 a + \dots$$
with an identity operator as a leading term.

We see that the algebra of observables in deformation quantization is extremely simple: we do not need a Hilbert space as in the traditional operator approach, we do not need to care about domains of unbounded operators. Even comparing to the C^*-algebra approach we see that the C^*-algebra structure is much more delicate one. Moreover, we don't need to worry how to construct the star-product: the Kontsevich theorem gives a universal construction starting with an arbitrary classical system. In contrast, other approaches may require quite non-trivial constructions (group representations, the theory of constrained systems).

2. Reduction

The algebra of quantum observables in deformation quantization may be treated as an extension of the algebra of classical observables which serve as leading terms. So, any construction in the former automatically descends to the latter. One can ask if this principle admits a reversion. In other words, can we lift any construction in the algebra of classical observables to the quantum level in the framework of deformation quantization. Of course, there is no rigorous theorem of such sort, this assertion may be viewed as an heuristic principle which explains the flexibility of deformation quantization. Let me illustrate this on the example of the symplectic reduction.

Consider a symplectic manifold (M, ω) with a Hamiltonian action of a compact connected and simply connected Lie group G. "Hamiltonian" here means that there exists the moment map $\mu : M \to g^*$. We assume, as usual, that zero is a non-critical value of the moment map and G act freely on the zero level manifold. Then the classical reduction theorem asserts that the manifold
$$B = M//G := \mu^{-1}(0)/G$$
is a symplectic manifold with a symplectic form ω_B such that its pull-back to $\mu^{-1}(0)$ coincides with the restriction of ω to $\mu^{-1}(0)$. The reduced algebra A^G/μ of

G-invariant functions modulo the ideal generated by μ coincides with the algebra $C^\infty(B)$ with a Poisson structure defined by the symplectic form ω_B.

What happens in deformation quantization? First of all, there exists a canonical construction of a deformation quantization on a symplectic manifold parametrized by a closed two-form

$$\Omega = \frac{i}{h}\omega + \omega_0 + h\omega_1 + \dots,$$

called the Weyl curvature. Clearly, it may be chosen G-invariant if the form Ω is invariant. Let $*_M$ and $*_B$ be the canonical star-products on M and B with the Weyl curvatures

$$\Omega_M = \frac{i}{h}\omega_M, \quad \Omega_B = \frac{i}{h}\omega_B$$

respectively. Then we have a reduction theorem in deformation quantization which sounds almost identically with the classical one, namely:

THEOREM 2.1. *The algebra* $(A(B) := C^\infty(B)[[h]], *_B)$ *is isomorphic to the quantum reduced algebra, which is the quotient of the subalgebra of* G*-invariants* $(A(M), *_M)^G$ *modulo the* $*$*-ideal generated by the moment map* μ.

This is a statement of the type "reduction commutes with quantization", known in geometric quantization as the Guillemin–Sternberg conjecture. While in the geometric quantization framework it is a serious theorem, for deformation quantization it is much more simple. Besides, it gives a natural correspondence between original and reduced observables.

3. Non-trivial coefficients

Deformation quantization admits a natural generalization to the case of non-trivial coefficients at least in the case of a symplectic manifold. For example, the coefficients may be matrices and they may form a non-trivial bundle over M, i.e., we may consider a deformation quantization with a coefficient bundle $K = \mathrm{Hom}\,(E, E)$ for some vector bundle E over M. Moreover, in practice we need even more complicated bundles. For instance, for the reduction theorem we need a bundle over the base B whose fibers are quantum observables on T^*G. The changes are as follows. The algebra of classical observables is now a non-commutative algebra of sections of the coefficient bundle $C^\infty(M, K)$, and we replace the condition 3 of Definition 1.1 by

$$C_1(a, b) = -\frac{ih}{2}\omega^{ij}\,\partial_i a\,\partial_j b$$

where ∂ now means a covariant derivative in the coefficient bundle.

A canonical construction of deformation quantization with non-trivial coefficients is quite general. It is meaningful even in infinite-dimensional setting of moduli spaces of connections on a Riemann surface, leading to a quantization procedure commuting with reduction.

4. Trace and index

Besides the algebra of observables, a mechanical system (classical or quantum) is characterized by its states: linear functionals on the algebra of observables with some further additional properties. Usually one imposes a kind of positivity. We will not go deeper into the subject, instead we consider a trace functional, which may be introduced without positivity.

DEFINITION 4.1. A *trace* is a linear functional

$$\mathrm{Tr} : A_c \to \mathbb{C}[[h]]h^{-n}$$

having the property

$$\mathrm{Tr}\,(a * b) = \mathrm{Tr}\,(b * a).$$

Here $A_c \in A$ is an ideal of compactly supported observables and at the right stands the space of Laurent formal series in h with negative powers not exceeding $n = (1/2)\dim M$.

A trace functional gives rise to a very interesting generalisation of the Atiyah–Singer index theorem for elliptic operators. In the simplest case when M is a compact symplectic manifold and the coefficient bundle is $\mathrm{Hom}\,(E, E)$, the index theorem consists in an explicit formula for the trace of the unit element:

$$\mathrm{Tr}\,1 = \int_M \mathrm{ch}\,E \exp\left(\frac{1}{2\pi h}\right)\widehat{A}(M). \tag{4.1}$$

Deformation quantization provides the most natural environment for the index theorem and its various modifications. In the context of deformation quantization they acquire quite different meaning than for elliptic operators (we will discuss this subject in the next section).

The theory of traces in symplectic case is rather dull: there exists a unique trace up to a normalization "constant". In contrast, the theory of traces for Poisson manifolds is more rich. The traces are parametrized by volume forms dV such that the Poisson tensor α is divergence-free. For each such volume form we have an index formula similar to (4.1) with the replacement of $\exp(\omega/2\pi h)$ by $i(\exp(\alpha/2\pi h))dV$.

I hope enough was said about the merits of deformation quantization. Concluding the first part of my talk, I would like to stress once more the heuristic principle that explains the merits of deformation quantization: any construction in the algebra of classical observables can be lifted to the algebra of quantum observables in the deformation quantization framework.

5. Difficulties with calculations

Now I am passing to the second part which can be titled "The drawbacks". To begin with, let us note that in practice h is a small positive number, proportional to the Planck constant with coefficients depending on physical parameters of the system. Thus, for calculations we need to substitute this numerical value of h into a formal power series. This operation is quite unclear except for the case when the series terminates and we have a polynomial. Moreover, we know from traditional quantum mechanics that physical parameters sometimes have discrete spectrum and sometimes not. Is it possible to explain this in the deformation quantization framework?

A reasonable way out is to treat formal power series in h as asymptotic ones, at least as far as calculations are concerned. In fact, in many cases one can interpret these series as asymptotic ones obtained by stationary phase method or the WKB method for differential equations. But in general it is unclear if we can interpret formal power series as asymptotic and in what sense. So, to my mind, most computational algorithms should begin like this: "Suppose we have a representation of our deformation business such that the formal power series in question are in

fact asymptotic ones in such and such sense. Then ...", and a necessary condition follows for such a representation to exist.

Let me illustrate the situation on the example of the index theorem. Suppose we have a representation of the algebra of quantum observables by means of operators in a Hilbert space such that the formal power series for $\operatorname{Tr} a$ is an asymptotic expansion of the operator trace at $h \to 0$ over some set $\Lambda \subset (0,1]$. Then all the indices must be integer-valued on the admissible set Λ modulo $O(h^\infty)$. Note that no computational difficulties arise for indices since they are given by finite series. One can show that in order to have integer-valued indices it is necessary and sufficient that

$$\frac{\omega}{2\pi h} + \frac{1}{2} c_1(T_{\mathbb{C}}(M)) \in H^2(M, \mathbb{Z}), \tag{5.1}$$

that is, we have recovered the quantization condition of Maslov's asymptotic quantization.

A similar application of the index theorem for elliptic operators we can see even for a classical case of the Dirac operator. In this case the index is given by the Atiyah-Singer formula

$$\operatorname{ind} D = \int_X \operatorname{ch} E \, \widehat{A}(X).$$

Surprisingly, the right-hand side is not always integer and its integrality is a necessary condition for the existence of the spin structure.

6. Is there a Spectral Theory?

We all know about the important role that the spectral theory for operators in Hilbert space plays in traditional quantum mechanics. In fact, it is responsible for the variety of spectra. But in deformation quantization there is nothing which may be compared in power with the spectral theorem though some separate notions one can easily define. For example, if $E \in \mathbb{R}$ is a non-critical level of the real Hamiltonian $H \in C^\infty(M)$ we can construct an eigenstate up to a normalizing factor as a non-trivial functional on the algebra of observables vanishing on the subspace $(H - E) * u$, $u \in A$. However, still we have problems

(1) how to treat critical values
(2) how to pull the eigenstates (critical and non-critical) together to obtain a holomorphic calculus of quantum observables.

To my mind, a spectral theory for deformation quantization should be developed, but maybe in a quite different way than for operators. It should incorporate Morse theory (or rather the Floer theory) in the spirit of Witten, as well as the index theory.

To conclude this section, let us consider an example where the spectrum may be completely determined by means of reduction and the index theorem. So, consider the quantum harmonic oscillator $H = |x|^2/2 \in C^\infty(\mathbb{R}^{2(n+1)})$. For any $E > 0$ the level manifold $H = E$ is a sphere $|x|^2 = 2E$ and the orbit space is $\mathbb{CP}^n$ with the symplectic form $\omega = 2E\omega_{FS}$ where ω_{FS} is the standard Fubini–Studi form on $\mathbb{CP}^n$. The quantization condition (5.1) yields

$$E = \pi h \left(k + \frac{n+1}{2} \right), \quad k \in \mathbb{Z}, \tag{6.1}$$

besides,

$$\operatorname{Tr} 1 = C^n_{E/\pi h+n-1}.$$

The the usual magic words follow: "Suppose that the reduced algebra, that is $A(\mathbb{CP}^n)$, corresponds to the algebra of operators in the eigenspace of the "operator" H with the eigenvalue E. Then the possible spectrum of H is given by (6.1) provided that $\operatorname{Tr} 1$ is a positive integer". On the set (6.1) we have $\operatorname{Tr} 1 = C^n_{n+k}$ which is greater than zero for $k \geq 0$. Thus, we indeed have found the spectrum with its multiplicities. Of course, one needs to give more precise meaning to the word "corresponds".

7. What is the Schrödinger Evolution?

The original idea of F. Bayen *et al.* was to get an alternative approach to quantum mechanics based on deformation quantization rather than on the operator theory. Instead of the Schrödinger evolution operator $\exp(iHt/h)$ they introduced the so-called star-exponential $\exp_*(iHt/h)$ which one can imagine intuitively as a series

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{it}{h}\right)^k \underbrace{H * H * \cdots * H}_{k \text{ times}}$$

or as a solution of the differential equation

$$\frac{\partial U}{\partial t} = \frac{i}{h} H * U, \quad U(0) = 1.$$

The idea was that the Fourier–Dirichlet series determines completely the spectrum of H as the set of frequencies with spectral projectors as coefficients. Unfortunately, this heuristic notion may be justified only in exceptional cases like the harmonic oscillator. On the other hand, we badly need such a notion, for example to define characters for group action. One of the possible ways to define it rigorously is to interpret it as a distribution $\operatorname{Tr} U(t)a$ where $a \in C_c(M)$ with $t \in [0, T]$ fixed. It turns out that under natural additional conditions the support of this distribution should coincide with fixed-point set of the classical hamiltonian flow. The contribution of the given fixed-point component also may be found up to a "constant" factor. But examples show that we need to extend the "constants" by rapidly oscillating exponents, understood as formal expressions. A problem arises of consistent choice of rapidly oscillating factors for each fixed-point component. Besides, the fixed-point set depends on t rather chaotically: the components may fuse and split. Thus, even if we succeed in definition of the star-exponential, one could hardly expect to extract from it an information about spectra. On the contrary, we rather could expect that the geometry of fixed-point components and the oscillating factors are governed by the quantization conditions. In any case, it would be interesting to understand the connection between quantization conditions and the behavior of fixed-point components.

In conclusion, summarizing the merits and drawbacks of deformation quantization, we see that the main difficulties originate from calculations, as a rule they require some artificial tricks. On the other hand, as long as we deal with constructions in the algebra of observables, deformation quantization is a flexible tool and gives us the maximal freedom.

MAX-PLANCK-INSTITUT FÜR MATHEMATIK, P.O.BOX: 7280, D-53072 BONN, GERMANY
E-mail address: fedosov@mpim-bonn.mpg.de

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Quantization as a functor

N. P. Landsman

ABSTRACT. Notwithstanding known obstructions to this idea, we formulate an attempt to turn quantization into a functorial procedure. We define a category Poisson of Poisson manifolds, whose objects are integrable Poisson manifolds and whose arrows are isomorphism classes of regular Weinstein dual pairs; it follows that identity arrows are symplectic groupoids, and that two objects are isomorphic in Poisson iff they are Morita equivalent in the sense of P. Xu. It has a subcategory LPoisson that has duals of integrable Lie algebroids as objects and cotangent bundles as arrows. We argue that naive C^*-algebraic quantization should be functorial from LPoisson to the well-known category KK, whose objects are separable C^*-algebras, and whose arrows are Kasparov's KK-groups. This limited functoriality of quantization would already imply the Atiyah–Singer index theorem, as well as its far-reaching generalizations developed by Connes and others. In the category KK, isomorphism of objects implies isomorphism of K-theory groups, so that the functoriality of quantization on all of Poisson would imply that Morita equivalent Poisson algebras are quantized by C^*-algebras with isomorphic K-theories. Finally, we argue that the correct codomain for the possible functoriality of quantization is the category RKK(I), which takes the deformation aspect of quantization into account.

> "First quantization is a mystery, but second quantization is a functor" (E. Nelson)
>
> "Comme l'on sait la "quantification géometrique" consiste à rechercher un certain foncteur de la catégorie des variétes symplectiques et symplectomorphismes dans celle des espaces de Hilbert complexes et des transformations unitaires (...) Il est bien connu qu'un tel foncteur n'existe pas" (from A. Crumeyrolle's review **MR81g:58016** of [**19**])

1. Introduction

The functoriality of second quantization (a construction involving exponential Hilbert spaces or Fock spaces) mentioned in our opening quote of Nelson is an almost trivial matter. The deep problem suggested by this quote is the possible

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functoriality of "first" quantization, which simply means the quantization of Poisson manifolds P.

The simplest example is probably $P = T^*(\mathbb{R}^n)$ with the usual Poisson structure. Defining quantization either by the Schrödinger representation $U_\hbar^S$ of the Heisenberg group H_n in dimension $2n$, or by the Weyl–Moyal prescription $Q_\hbar^W$ (which points of view are essentially equivalent), it follows either way that the quantization of $T^*(\mathbb{R}^n)$ is functorial with respect to affine linear symplectomorphisms and unitary intertwiners; see, e.g., [**18**] or [**32**]. Taking Weyl–Moyal quantization to be concrete, this statement specifically means that one has

$$Q_\hbar^W(f \circ L^{-1}) = U_\hbar^M(L) Q_\hbar^W(f) U_\hbar^M(L)^*, \tag{1}$$

where $f \in C_c^\infty(T^*(\mathbb{R}^n))$ (for simplicity), L is an affine linear symplectomorphism, and $U_\hbar^M$ is the representation of the affine symplectic group composed of the metaplectic representation of the linear symplectic group and the (projective) Schrödinger representation $U_\hbar^S$ of the translation group in dimension 2n. As is well known, the Groenewold–Van Hove theorem (cf. [**20, 21**] for an up-to-date treatment) precludes functoriality under a larger class of classical transformations [**19**]. This seems about all that is known about the functoriality of (first) quantization.

The above example has a number of instructive features. Firstly, $T^*(\mathbb{R}^n)$ has a large amount of symmetry, which is fully exploited by the Weyl–Moyal quantization prescription. The rather meager functoriality properties are a direct consequence of this symmetry. Indeed, the Berezin–Toeplitz quantization prescription on $T^*(\mathbb{R}^n)$ (relying on its Kähler structure), which is physically as acceptable as the Weyl–Moyal prescription, and is much better behaved analytically [**32**], enjoys even less functoriality. Since both prescriptions hinge on rather special properties of $T^*(\mathbb{R}^n)$, for the sake of generalization it would seem wise not to let the notion of functoriality of quantization rely on the precise details of a quantization prescription, but rather on a certain equivalence class to which it belongs.

Secondly, the Groenewold–Van Hove no-go theorem suggests that taking unitary transformations on the quantum side does not leave enough room to manoeuvre in the codomain category of a potential quantization functor. Hence one needs a different class of arrows at least in the quantum category. It is convenient to work with C^*-algebras rather than concrete Hilbert spaces; instead of unitary operators one should then speak of *-automorphisms. For example, eq. (1) defines conjugation by $U_\hbar^M(L)$ as a *-automorphism of the C^*-algebra of compact operators on $L^2(\mathbb{R}^n)$.

Furthermore, it is extremely unnatural to only work with simple C^*-algebras (like the compact operators), and once one has decided to work with general C^*-algebras, it goes without saying that one should consider general Poisson manifolds, instead of merely symplectic ones. The conclusion so far, then, is that the naive idea that quantization ought to be functorial with respect to isomorphisms of Poisson manifolds and *-isomorphisms of C^*-algebras, let alone the stronger requirement of functoriality with respect to Poisson maps and *-homomorphisms, respectively, has to be given up.

More suitable categories of C^*-algebras necessarily have a weaker notion of isomorphism than *-isomorphism. To obtain powerful results, and also to restore a certain parallel between the classical and the quantum categories, we will accordingly use a classical category in which isomorphism of objects is weaker than isomorphism of Poisson manifolds in the usual sense.

Our original idea was that one should use Morita equivalence on both sides; for Poisson manifolds this notion was developed by Xu [**59**], whereas the older C^*-algebraic theory is due to Rieffel [**51, 52**]. If one has categories in which isomorphism of objects comes down to Morita equivalence, then the possible functoriality of quantization would imply that quantization preserves Morita equivalence. Such categories are easily defined [**35, 36**]. On the classical side one has the category Poisson, whose objects are integrable Poisson manifolds, and whose arrows are isomorphism classes of regular Weinstein dual pairs. On the quantum side one has a category C* whose objects are C^*-algebras and whose arrows are unitary equivalence classes of Hilbert bimodules (for the latter see also [**56, 15**]).

Apart from the (flawed) idea in the previous paragraph, the use of the categories Poisson and C* was in addition motivated by the fact that the possible functoriality of quantization as a map from Poisson to C* would imply the "quantization commutes with reduction" principle (see [**32**] and references therein). Hence this perhaps somewhat mysterious principle would appear in a canonical mathematical light. For let $Q \leftarrow S_1 \rightarrow P$ and $P \leftarrow S_2 \rightarrow R$ be regular Weinstein dual pairs, quantized by an A-B Hilbert bimodule $\mathcal{Q}(S_1)$ and a B-C Hilbert bimodule $\mathcal{Q}(S_2)$, respectively. Here $B_0 \cong C_0(P)$. Functoriality of quantization implies

$$\mathcal{Q}(S_1 \circledcirc_P S_2) = \mathcal{Q}(S_1) \hat{\otimes}_B \mathcal{Q}(S_2).$$

Now composition of arrows $\circledcirc$ in Poisson is given by symplectic reduction, whereas the interior tensor product of Rieffel that defines arrow composition $\hat{\otimes}$ in C* is a quantized version of the classical reduction procedure [**31, 32, 35**]. Hence the left-hand side is "quantization after reduction," whereas the right-hand side stands for "reduction after quantization."

As will be recalled below, Poisson has a subcategory LPoisson whose objects are duals of integrable Lie algebroids, and whose arrows are cotangent bundles. The point is that there indeed exists a functor from LPoisson to C* resembling quantization on the object side, so that Morita equivalent Poisson manifolds in LPoisson are quantized by Morita equivalent C^*-algebras. Although this is a nontrivial result, it was pointed out to the author by Alan Weinstein that it is rather untypical, and that it would be mistake to conjecture an extension of this result to all (integrable) Poisson manifolds (as we did in a previous draft of this paper, of which Weinstein was the referee). He actually pointed out a class of counterexamples, as follows. Take any two tori of the same (even) dimension, but carrying different symplectic structures. These will always be Morita equivalent as Poisson manifolds [**59**]. However, one can easily choose the symplectic structures in such a way that their respective quantizations (as defined in [**53, 54**]) fail to be Morita equivalent as C^*-algebras (for any value of $\hbar$); cf. [**55**].

These counterexamples show that Morita equivalence is still not coarse enough on the quantum side to be preserved by quantization, and suggests that it might be more appropriate to use K-equivalence of C^*-algebras (i.e., isomorphism of K-groups). The natural codomain of a possible quantization functor is then clearly the category KK, whose objects are separable C^*-algebras, and whose arrows are Kasparov's KK-groups [**3, 26**]. Although isomorphism of objects in KK is not the same as isomorphism of their K-groups, the latter is implied by the former, and the category KK has the enormous computational advantage (for example, over C*) that the Hom-spaces $KK(A, B)$ are abelian groups. The results in [**36**] and [**24**] then strongly suggest that quantization should be functorial from LPoisson to KK.

It should be mentioned that even this limited functoriality of quantization would already imply the Atiyah–Singer index theorem as well as its generalization to foliations due to Connes [**8, 10**]. Moreover, it further motivates the generalization of the latter to a general index theorem for Lie groupoids called for in [**37**]. One can only marvel at the possible implications that the complete functoriality of quantization would have.

The use of KK is still unsatisfactory, in that its objects are single C^*-algebras; one effectively works at some fixed value of Planck's constant $\hbar$ (like in geometric quantization). In the spirit of deformation quantization, it is much better to use continuous fields of C^*-algebras as the target of the quantization operation, as first proposed by Rieffel [**53**]. This suggests the use of the category RKK(I) as the codomain of a possible quantization functor. For technical reasons this category has upper semicontinuous fields of separable C^*-algebras over the interval $I = [0, 1]$ (seen as the parameter space of $\hbar$) as objects, and the so-called representable KK-groups $\mathcal{R}KK(I, -, -)$ [**27**] as arrows.

The plan of this paper is as follows. In Section 2 we review the construction of the "classical" category Poisson [**35, 36**], which is the domain of the alleged quantization functor in any of the approaches we discuss. Section 3 describes the simplest candidate C* for the codomain category of this functor [**15, 35, 56**]. This category also lies at the basis of the construction of the more sophisticated codomains used further on. In Section 4 we prove that quantization is functorial from the subcategory LPoisson of Poisson to C*. Section 5 recalls Kasparov's category KK [**3, 26**], and refines the previous result so as to apply to KK rather than C*. In Section 6 we turn to deformation quantization. In particular, we indicate how the original ideas of formal deformation quantization [**2**] can be realized in the context of C^*-algebras, so as to motivate both Rieffel's axioms for C^*-algebraic quantization [**53**] and the author's modification of these [**32**]. As outlined in Section 7, these considerations immediately lead to a refinement C*(I) of the category C*, in such a way that the step from C* to KK is analogous to the passage from C*(I) to a category RKK(I). The latter, then, is our proposal for the codomain of a potential functorial quantization procedure. Thus quantization should be a functor from Poisson to RKK(I).

Finally, let us note that, in view of the audience towards which these lectures were directed, some definitions are given in greater detail than others. Most participants were familiar with the likes of derived categories and symplectic groupoids, whereas elementary knowledge of operator algebras seemed lacking.

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criticized by Alan Weinstein, whose comments were instrumental in reaching the point of view expressed in the present version.

2. The classical category

We recall the definition of the category of integrable Poisson manifolds introduced in [**35, 36**]. This category relies on the theory of symplectic groupoids (cf. [**7, 11**] and refs. therein, as well as the forthcoming 2nd edition of [**40**]). The objects in Poisson satisfy the following condition.

DEFINITION 1. *A Poisson manifold P is called integrable when there exists a symplectic groupoid over P.*

This definition is due to [**11**]. The integrability assumption is necessary in order to have identities in Poisson; see below. In this paper we assume for simplicity that the symplectic groupoid in question is Hausdorff. This assumption can be dropped at the expense of considerable technical complications, which can be overcome if the definition of continuous or smooth functions on non-Hausdorff manifolds in [**13**] is used.

The arrows in Poisson are isomorphism classes of certain Weinstein dual pairs. Recall that, given two Poisson manifolds P and Q, a Weinstein dual pair $Q \leftarrow S \rightarrow P$, simply called a dual pair in what follows, consists of a symplectic manifold S and Poisson maps $q : S \rightarrow Q$ and $p : S \rightarrow P^-$, such that $\{q^*f, p^*g\} = 0$ for all $f \in C^\infty(Q)$ and $g \in C^\infty(P)$ [**25, 58**]. Two Q-P dual pairs $Q \xleftarrow{q_i} \tilde{S}_i \xrightarrow{p_i} P$, $i = 1, 2$, are isomorphic when there is a symplectomorphism $\varphi : \tilde{S}_1 \rightarrow \tilde{S}_2$ for which $q_2\varphi = q_1$ and $p_2\varphi = p_1$.

The notion of regularity for dual pairs is explained in [**35, 36**]; its goal is to guarantee the existence of the following symplectic quotients. Part of the regularity condition is the stipulation that the maps p and q be complete, and that q is a surjective submersion. Let R be a third integrable Poisson manifold, and let $Q \leftarrow S_1 \rightarrow P$ and $P \leftarrow S_2 \rightarrow R$ be regular dual pairs. The embedding $S_1 \times_P S_2 \subset S_1 \times S_2$ is coisotropic [**32**]; we denote the corresponding symplectic quotient by $S_1 \circledcirc_P S_2$. This is the middle space of a regular dual pair $P \leftarrow S_1 \circledcirc_P S_2 \rightarrow R$. The operation $\circledcirc$ is associative up to isomorphism.

For suitable choices of dual pairs, the product $\circledcirc$ is the same as Marsden–Weinstein reduction [**32**]; this should not be surprising in view of its general definition in terms of symplectic reduction.

Using results in [**43**] and [**41**], it can be shown that if P is integrable, then there exists an s-connected and s-simply connected symplectic groupoid $\Gamma(P)$ whose base space is isomorphic to P as a Poisson manifold. Moreover, $\Gamma(P)$ is unique up to isomorphism of symplectic groupoids. Cf. Lemma 5.6 in [**35**]. The upshot of this is that the isomorphism class $[P \xleftarrow{t} \Gamma(P) \xrightarrow{s} P]$ is a two-sided identity for $\circledcirc_P$. (We denote the source and target maps in a groupoid by s and t, respectively.)

DEFINITION 2. *The category Poisson has integrable Poisson manifolds as objects, and isomorphism classes of regular dual pairs as arrows.*

The original reason for the introduction of this category was the fact that two Poisson manifolds are Morita equivalent in the sense of Xu [**59**] iff they are isomorphic objects in Poisson; see Prop. 5.13 in [**35**]. In particular, a Poisson manifold is integrable iff it is Morita equivalent to itself (as already observed by

Xu). The category Poisson is a classical analogue of the category of C^*-algebras with unitary equivalence classes of Hilbert bimodules as arrows [**35, 36**].

We now introduce a subcategory LPoisson of Poisson on which we will be able to define a quantization functor. This subcategory is not full, though in an informal sense it is large and interesting. Recall that a Lie groupoid G over G_0 has an associated Lie algebroid $A(G)$, which is a vector bundle over G_0 [**40**]. The dual vector bundle $A^*(G)$ is equipped with a canonical Poisson structure [**11, 12**] (also cf. [**11, 32**] for a review). This Poisson structure is linear, in that the Poisson bracket of two (fiberwise) linear functions is again linear. Conversely, any linear Poisson structure is the dual of some Lie algebroid [**12**] (but this Lie algebroid need not be integrable). Poisson manifolds of the form $A^*(G)$ include all cotangent bundles, all duals of Lie algebras, all manifolds with zero Poisson bracket, all semidirect product Poisson structures, and all Poisson manifolds defined by a foliation.

The objects of LPoisson are the Poisson manifolds $A^*(G)$ associated to arbitrary Lie groupoids G. The arrows in LPoisson are isomorphism classes of regular dual pairs that are of the following form. Let G and H be Lie groupoids, and suppose that a manifold M is a G-H bibundle; we write $G \rightarrowtail M \leftarrowtail H$. This means that G and H act smoothly on M on the left and on the right, respectively, in such a way that the actions commute; cf. [**35, 42, 44, 45**]. A construction in [**34**], generalizing the momentum map of symplectic geometry from Lie groups to Lie groupoids, associates a dual pair

$$A^*(G) \xleftarrow{J_L} T^*(M) \xrightarrow{J_R} A^*(H) \tag{2}$$

to such a bibundle. For a dual pair of this form to be regular, it suffices that the bibundle be principal [**42, 44, 45**] (also see [**35**] for a review); this means that the base map $\pi : M \to G_0$ of the G-action on M is a surjective submersion, and that H acts freely and transitively on the fibers of π. It follows that $M/H \cong G_0$. In foliation theory principal bibundles are seen as generalized maps between leaf spaces (see, e.g., [**24, 44**]), and, more generally, principal bibundles are sometimes called generalized maps between groupoids.

For example, the canonical G-G bibundle $G \rightarrowtail G \leftarrowtail G$ gives rise to the dual pair

$$A^*(G) \xleftarrow{t} T^*(G) \xrightarrow{s} A^*(G), \tag{3}$$

where $T^*(G)$ is the cotangent bundle defined in [**11**]. This is precisely the symplectic groupoid associated to the Poisson manifold $A^*(G)$.

Let LG be the category of Lie groupoids, whose arrows are isomorphism classes of principal bibundles (see [**35, 44, 45**]). Composition of arrows is defined as follows. Suppose one has principal bibundles $G \rightarrowtail M \leftarrowtail H$ and $H \rightarrowtail N \leftarrowtail K$. The fiber product $M \times_{H_0} N$ carries a right H action, given by $h : (m, n) \mapsto (mh, h^{-1}n)$ (defined as appropriate). The orbit space

$$M \circledast_H N = (M \times_H N)/H \tag{4}$$

is a G-K bibundle in the obvious way. This defines a product on matched bibundles, which becomes associative on isomorphism classes. We define $\mathsf{L\tilde{G}}$ as the full subcategory of LG whose objects are s-connected and s-simply connected Lie groupoids.

According to Thm. 3 and eq. (4.30) in [**36**], the above procedure defines a functor A^* from $\mathsf{L\tilde{G}}$ to Poisson. That is, on objects one has $G \mapsto A^*(G)$, whereas

on arrows the functor in question maps the isomorphism class of a G-H bibundle M to the isomorphism class of the dual pair (2). The operation A^* may also be defined on LG, but it fails to be functorial because identities are not always preserved. Note that A^* indeed maps $\mathsf{L}\tilde{\mathsf{G}}$ into Poisson: $A^*(G)$ is actually integrable, with associated symplectic groupoid $T^*(G)$.

DEFINITION 3. *The category* LPoisson *is the image of the functor*

$$A^* : \mathsf{L}\tilde{\mathsf{G}} \to \mathsf{Poisson}.$$

Thus LPoisson has Poisson manifolds of the form $A^*(G)$, where G is a Lie groupoid, as objects, and isomorphism classes of cotangent bundles of the form (2), where M is a principal bibundle, as arrows. Note that LPoisson contains all identities as appropriate, since the symplectic groupoid $T^*(G)$ is s-connected and s-simply connected whenever G is.

3. The simplest quantum category

Within the Hilbert space formalism for quantum mechanics, it is natural to assume that the observables of a quantum system form a C^*-algebra [**22, 32**]. Recall that a C^*-algebra A is a complex associative algebra with involution, equipped with the structure of a Banach space, such that $\|ab\| \leq \|a\|\|b\|$ and $\|a^*a\| = \|a\|^2$ for all $a, b \in A$. A C^*-algebra can always be faithfully represented as a norm-closed involutive algebra of bounded operators on a Hilbert space, on which the involution is just the adjoint, and the norm is the usual operator norm. See [**9**] for an overview of the use of C^*-algebras in modern mathematics.

At some fixed value of Planck's constant $\hbar$, quantization then associates a C^*-algebra to a Poisson algebra. This view of quantization is not satisfactory, and ought to be replaced by the idea of deformation quantization, but for the moment we stick to it for pedagogical reasons. Thus, to a first approximation, the objects of the codomain category of a possible quantization functor should be C^*-algebras. In order to assemble these into a category, the most obvious choice would be to take the arrows to be *-homomorphisms, but the pertinent isomorphisms would then be *-isomorphisms. As mentioned in the Introduction, this choice is inappropriate for quantization theory. A more suitable class of arrows between C^*-algebras is formed by (isomorphism classes of) so-called Hilbert bimodules.

A Hilbert bimodule is the C^*-algebraic analogue of a bimodule for algebras over a given ring [**51, 52**]. A new feature compared to the purely algebraic situation is that an A-B Hilbert bimodule is endowed with a B-valued inner product. The complete definition is as follows.

DEFINITION 4. *Let A and B be C^*-algebras. An A-B Hilbert bimodule is an A-B bimodule E (where A and B are seen as complex algebras, so that E is a complex linear space) with a compatible B-valued inner product. Thus there is a sesquilinear map $\langle\,,\,\rangle : E \times E \to B$, linear in the second and antilinear in the first entry, satisfying $\langle x, y\rangle^* = \langle y, x\rangle$, $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ iff $x = 0$. The compatibility of the inner product with the remaining structures means that firstly E has to be complete in the norm $\|x\|^2 = \|\langle x, x\rangle\|$, secondly that $\langle x, yb\rangle = \langle x, y\rangle b$, and thirdly that $\langle a^*x, y\rangle = \langle x, ay\rangle$ for all $x, y \in E$, $b \in B$, and $a \in A$. Finally, the left action of A on E is required to be nondegenerate in the sense that AE is dense in E.*

Note that A-$\mathbb{C}$ Hilbert bimodules are just Hilbert spaces carrying a nondegenerate representation of A. On the other hand, a $\mathbb{C}$-B Hilbert bimodule under the obvious action of $\mathbb{C}$ (by multiples of the unit operator) is called a Hilbert B module.

The basic example of an A-A Hilbert bimodule is $E = A$ with the obvious actions and the inner product $\langle a, b\rangle = a^*b$. See [**30**] for the basic theory, cf. [**35**] for a comparison between Hilbert bimodules and analogous structures in mathematics, and have a look at [**32**] for the use of Hilbert bimodule in quantization theory. One feature of algebraic bimodules that survives in the Hilbert case is the existence of a bimodule tensor product [**51, 52**]: from an A-B Hilbert bimodule E and a B-C Hilbert bimodule $\tilde{E}$ one can form an A-C Hilbert bimodule $E\hat{\otimes}_B\tilde{E}$, called the interior tensor product of E and $\tilde{E}$. For the following definition [**15, 35, 56**] we also need the notion of unitary equivalence, which the reader may guess (see [**30**], p. 24).

DEFINITION 5. *The category* C^* *has* C^**-algebras as objects, and unitary equivalence classes of Hilbert bimodules as arrows. The arrows are composed by the interior tensor product.*

It follows that the identity arrow at A is the canonical Hilbert bimodule A defined above. In addition, it turns out that two C^*-algebras are Morita equivalent [**52**] iff they are isomorphic as objects in C^* [**15, 35, 56**]. This property suggests that the category C^* should be regarded as a quantum version of Poisson.

4. Functoriality of simple quantization

We are now in a position to state the first result on the functoriality of quantization.

THEOREM 1. *There exists a functor* $\mathcal{Q} : \mathsf{LPoisson} \to \mathsf{C}^*$ *that on objects maps* $A^*(G)$ *to* $C^*(G)$.

The reason why $C^*(G)$ may indeed be seen as the quantization of $A^*(G)$ at some fixed value of $\hbar$ is actually to be found in deformation quantization [**32, 33, 37, 49**].

Proof. The functor $\mathcal{Q}$ is the composition of the following functors:

$$\mathsf{LPoisson} \stackrel{(A^*)^{-1}}{\longrightarrow} \mathsf{L\tilde{G}} \stackrel{C^*}{\longrightarrow} \mathsf{C}^*(I). \tag{5}$$

We discuss the functors $(A^*)^{-1}$ and C^* in turn.

Firstly, it follows from Props. 3.3 and 3.5 in [**43**] that for s-connected and s-simply connected Lie groupoids G the association $G \mapsto A^*(G)$ is invertible. Hence $A^* : \mathsf{L\tilde{G}} \to \mathsf{LPoisson}$ is an isomorphism of categories, with inverse $(A^*)^{-1}$. (It would have been sufficient for $A^* : \mathsf{L\tilde{G}} \to \mathsf{LPoisson}$ to define an equivalence of categories, for even then it would possess an inverse up to natural isomorphism, which would be enough for our purposes.)

Secondly, on objects the map $G \mapsto C^*(G)$ from Lie groupoids to C^*-algebras is the well-known association of a convolution C^*-algebra to a Lie groupoid [**9**] (or, more generally, to a locally compact groupoid with Haar system [**50**]). Following a special case in [**46**] (in which the arrows were taken to be Morita equivalences of groupoids), the map $G \mapsto C^*(G)$ was extended to a functor from LG to the category C^* in [**36**].

It follows that $\mathcal{Q} = C^* \circ (A^*)^{-1} : \mathsf{LPoisson} \to \mathsf{C}^*$ is a functor. ∎

We can illustrate this result by noting that an identity (3) in LPoisson is mapped to the canonical $C^*(G)$-$C^*(G)$ Hilbert bimodule $C^*(G)$. Hence the symplectic manifold $T^*(G)$ and the Poisson manifold $A^*(G)$ are both mapped into the C^*-algebra $C^*(G)$, but $T^*(G)$ is seen as an arrow, and $A^*(G)$ is regarded as an object in LPoisson, so that the former is mapped to $C^*(G)$ as (the middle space of) a Hilbert bimodule, whereas the latter is sent to $C^*(G)$ seen as a C^*-algebra.

5. The category KK

As mentioned in the Introduction, the preceding theorem cannot be extended to all of Poisson, because noncommutative tori provide counterexamples. We therefore propose to replace the category C^* by an analogous category KK, in which at least these counterexamples are circumvented, and whose use is very attractive in many ways. Perhaps the main motivation for looking at quantization as a functor taking values in KK is that the well-known relationship between quantum mechanics and index theory [**16, 17, 57**] would be clarified by such functoriality.

The category KK emerged from Kasparov's work on K-theory [**26**], and is discussed in detail in [**3**]. Here we only sketch the main points that allow one to understand that KK is a subtle and deep modification of C^*. Given two separable C^*-algebras A and B, one defines $\mathbb{E}(A,B)$ as the collection of A-B Hilbert bimodules E that are countably generated in B, and are equipped with the following additional structure.

Firstly, E should be of the form $E = E_1 \oplus E_2$, where each E_i is an A-B Hilbert bimodule. Secondly, and this is the main feature, there should be an operator $F : E \to E$ that is adjointable (i.e., there is $F^* : E \to E$ such that $\langle F^*x, y\rangle = \langle x, Fy\rangle$) and odd (in that $F(E_1) \subseteq E_2$ and $F(E_2) \subseteq E_1$). This F should be an almost unitary intertwiner of $A \restriction E_1$ and $A \restriction E_2$, in that for each $a \in A$ the operators $[F,a]$, $(F^2-1)a$, and $(F-F^*)a$ be compact. (Here an operator on an A-B Hilbert bimodule E is said to be compact when it can be approximated in norm by linear combinations of rank one operators of the form $z \mapsto x\langle y,z\rangle$ for $x,y \in E$. In noncommutative geometry, compact operators are treated as infinitesimals [**9**].)

This setting was mainly motivated by index theory, in which $A = C(X)$ for some compact manifold X, $B = \mathbb{C}$, E is the Hilbert space of L^2-sections of some spinor bundle over X, and F is a pseudodifferential operator; see[**23**]. In general, the category C^* is enormously enriched by requiring the presence of F. However, when the A action on some A-B Hilbert bimodule E_1 happens to be by compact operators, one may choose $E = E_1 \oplus 0$ and $F = 0$ so as to obtain an element of $\mathbb{E}(A,B)$. Such elements may even survive the step to be explained next.

Elements (E,F) of $\mathbb{E}(A,B)$ are called Kasparov cycles. Elements of $KK(A,B)$ are equivalence classes of such cycles under the following relations: unitary equivalence, translation of F along norm-continuous path, and the addition of degenerate Kasparov cycles. The latter are those for which the operators $[F,a]$, $(F^2-1)a$, and $(F-F^*)a$ are 0 for all a. The ensuing equivalence relation may be reexpressed in a number of alternative forms [**3**].

It is not difficult to see that $KK(A,B)$ is an abelian group; the group operation is the direct sum of both bimodules and operators F, and the inverse of the class of a Kasparov cycle (E,F) is the class of $(E^{\mathrm{op}},-F)$ (where E^{op} is E with the opposite grading). Morover, the association $(A,B) \mapsto KK(A,B)$ is contravariantly functorial in the first entry, and covariantly functorial in the second. One recovers

K-theory as a special case of KK-theory by $K_0(A) \cong KK(\mathbb{C}, A)$, with topological K-theory as the special case $K^0(X) \cong K_0(C(X))$ whereas K-homology [**23**] emerges as $K^0(A) \cong KK(A, \mathbb{C})$.

The deepest aspect of Kasparov's theory is the existence of the so-called intersection product

$$KK(A, B) \times KK(B, C) \to KK(A, C),$$

which is functorial in all conceivable ways. This leads to the category KK, whose objects are separable C^*-algebras, and whose arrows are the KK-groups. More precisely, the Hom-space of arrows from B to A is $KK(A, B)$. In addition, $KK(A, B)$ defines a space of homomorphisms from $K_0(A)$ to $K_0(B)$ through the intersection product

$$KK(\mathbb{C}, A) \times KK(A, B) \to KK(\mathbb{C}, B) \cong K_0(B).$$

In particular, if two C^*-algebras are isomorphic in KK, then their K-groups are isomorphic.

Refining Theorem 1, we now conjecture that there exists a functor from Poisson to KK that on objects maps $A^*(G)$ to $C^*(G)$. The evidence for this idea comes from the noncommutative geometry approach to index theory [**8, 9, 10**], as follows. The crucial step in the proof of the K-theoretic version of the Atiyah–Singer index theorem [**1**] is the association of a Gysin or wrong-way map $f! : K^0(X) \to K^0(Y)$ to a continuous (and usually smooth) map $f : X \to Y$ between locally compact spaces (usually manifolds) X and Y. For example, an embedding $M \hookrightarrow \mathbb{R}^n$ with pullback $T^*(M) \hookrightarrow \mathbb{R}^{2n}$ induces a map $K^0(T^*(M)) \to K^0(\mathbb{R}^{2n}) \cong \mathbb{Z}$, which is the topological index of Atiyah and Singer [**1**]. In KK-theory, $f!$ is seen as an element of $KK(C_0(X), C_0(Y))$, inducing the map between $K^0(X)$ and $K^0(Y)$ through the intersection product as explained above. The proof of the index theorem hinges on the property

$$(g \circ f)! = f! g!, \tag{6}$$

where the right-hand side is given by the intersection product.

To generalize this, it is useful to regard a space X as a groupoid (in which X is both the base and the total space of the groupoid, and the source, target, and object inclusion maps are all equal to the identity map), so that $C_0(X)$ is the C^*-algebra $C^*(X)$ of the groupoid X. One may then attempt to generalize the setting of the preceding paragraph to construct a functor from LG to KK. In other words, an object G is mapped into $C^*(G)$, and a principal G-H bibundle, which we now call F with some abuse of notation, is mapped into $f! \in KK(C^*(G), C^*(H))$. Then (6) is to hold, along with the preservation of identities.

For the longitudinal index theorem for foliations [**8, 9, 10**] it is sufficient to do this for the case that G is a space X and H is the holonomy groupoid of a foliation. The more symmetric case that G and H are both holonomy groupoids was treated in [**24**]; in both situations one has to impose an additional technical condition (of K-orientability) on F. The case that G and H are both arbitrary Lie groupoids has not been treated yet in the literature, but this should be possible. The ensuing functor from LG to KK could then be composed with the functor $(A^*)^{-1}$ from the proof of Theorem 1 to obtain the desired functor from LPoisson to KK. Our conjecture then asks for an extension of this functor from LPoisson to Poisson.

6. From formal to C^*-algebraic deformation quantization

As mentioned in the Introduction, the use of the category KK is still unsatisfactory. Its objects are single C^*-algebras, describing quantum-mechanical algebras of observables at some fixed value of $\hbar$. However, in the context of quantization theory it is important to study quantum theory for a range of values of Planck's "constant" $\hbar$, and to control the classical limit. This can be done in a purely algebraic way [**2**], or in an analytic C^*-algebraic way, as first proposed by Rieffel [**53**] (also cf. [**32**]).

We start with some remarks on the purely algebraic approach, called formal deformation quantization or star-product quantization, which serve the purpose of stressing the analogy between formal and C^*-algebraic deformation quantization.

A star-product on a Poisson manifold P endows the free module

$$C^\infty(P)[[\hbar]] = C^\infty(P,\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$$

over the commutative ring $\mathbb{C}[[\hbar]]$ of complex formal power series in one variable with the structure of an associative unital algebra over $\mathbb{C}[[\hbar]]$ (whose product is conventionally written as $*$), in such a way that

$$C^\infty(P)[[\hbar]]/\hbar C^\infty(P)[[\hbar]] \cong C^\infty(P)$$

as algebras over $\mathbb{C}$ (so that $f*g-g*f=0$ in $C^\infty(P)[[\hbar]]/\hbar C^\infty(P)[[\hbar]]$), and that Dirac's condition

$$f*g-g*f+i\hbar\{f,g\}=0$$

holds in $C^\infty(P)[[\hbar]]/\hbar^2 C^\infty(P)[[\hbar]]$. To state these axioms, it is crucial that there is a canonical map $f \mapsto f = f + 0\cdot\hbar + 0\cdot\hbar^2 + \cdots$ from $C^\infty(P)$ to $C^\infty(P)[[\hbar]]$. Now a unital algebra A over $\mathbb{C}[[\hbar]]$ is nothing but a $\mathbb{C}[[\hbar]]$-algebra A in the sense that there is an injective ring homomorphism from $\mathbb{C}[[\hbar]]$ into the center of A; cf. [**38**, p. 121]. Hence, one could define a generalized star-product on a Poisson manifold P as an associative unital $\mathbb{C}[[\hbar]]$-algebra A such that

(1) $A/\hbar A \cong C^\infty(P)$ as algebras over $\mathbb{C}$;

(2) there is a cross-section $Q : C^\infty(P) \to A$ of the canonical projection $\pi : A \to A/\hbar A$ for which Dirac's condition holds in the sense that

$$Q(f)*Q(g)-Q(g)*Q(f)+i\hbar Q(\{f,g\})=0$$

in $A/\hbar^2 A$.

Rieffel's analytic approach [**53**], based on the use of continuous fields of C^*-algebras, was a direct analogue of the original definition of a star-product, in that his fiber algebras $A_\hbar$ were obtained by putting an $\hbar$-dependent product $*_\hbar$ as well as an $\hbar$-dependent norm $\|\cdot\|_\hbar$ on $C^\infty(P)$ (assuming, for simplicity, that P is compact), and completing. Hence also here one has a canonical map $f \mapsto f$, this time from $C^\infty(P)$ to $A_\hbar$ (for each value of $\hbar$), in terms of which Rieffel formulated Dirac's condition as

$$\lim_{\hbar\to 0} \| \frac{i}{\hbar}(f *_\hbar g - g *_\hbar f) - \{f,g\}\|_\hbar = 0.$$

It was subsequently realized that more general continuous fields of C^*-algebras were needed in order to incorporate examples related to Berezin–Toeplitz quantization; cf. [**32**] and references therein. In the present context, such fields are best described using the formalism of $C(X)$ C^*-algebras, which we now recall.

The following definition is due to Kasparov [**27**] (in the more general case of locally compact X). We will only need the case $X = I$.

DEFINITION 6. *Let X be a compact Hausdorff space. A $C(X)$ C^*-algebra is a C^*-algebra A with a unital embedding of $C(X)$ in the center of its multiplier algebra. In other words, A comes equipped with a unital injective *-homomorphism $C(X) \to \mathcal{Z}(\mathcal{M}(A))$.*

The structure of $C(X)$ C^*-algebras was fully clarified by Nilsen [**47**], as follows. A field of C^*-algebras is a triple $(X, \{A_x\}_{x\in X}, A)$, where $\{A_x\}_{x\in X}$ is some family of C^*-algebras indexed by X, and A is a family of sections (that is, maps $f : X \to \coprod_{x\in X} A_x$ for which $f(x) \in A_x$) that is

(1) a C^*-algebra under pointwise operations and the natural norm

$$\|f\| = \sup_{x\in X} \|f(x)\|_{A_x};$$

(2) closed under multiplication by $C(X)$;
(3) full, in that for each $x \in X$ one has $\{f(x) \mid f \in \Gamma\} = A_x$.

The field is said to be continuous when for each $f \in A$ the function $x \mapsto \|f(x)\|$ is in $C(X)$ (this is equivalent to the corresponding definition of Dixmier [**14**]; cf. [**28**]). The field is upper semicontinuous when for each $f \in A$ and each $\varepsilon > 0$ the set $\{x \in X \mid \|f(x)\| \geq \varepsilon\}$ is compact.

Thm. 2.3 in [**47**] now states that a $C(X)$ C^*-algebra A defines a unique upper semicontinuous field of C^*-algebras

$$(X, \{A_x = A/C(X, x)A\}_{x\in X}, A).$$

Here

$$C(X, x) = \{f \in C(X) \mid f(x) = 0\},$$

and, with abuse of notation, $f \in A$ is identified with the section

$$x \mapsto \pi_x(f),$$

where $\pi_x : A \to A_x$ is the canonical projection.

Moreover, Blanchard [**5**] proved that a $C(X)$ C^*-algebra A defines a continuous field of C^*-algebras whenever the map $x \mapsto \|\pi_x(f)\|$ is continuous for each $f \in A$. Thus a continuous field of C^*-algebras over X may be described as a $C(X)$ C^*-algebra with this additional continuity condition.

It should be noted that, unlike vector bundles, continuous fields of C^*-algebras may well fail to be locally trivial. Here the restriction of a continuous field

$$(X, \{A_x\}_{x\in X}, A)$$

to some open subset $Y \subset X$ is said to be trivial when $A_x = B$ for all $x \in Y$, and A contains $C_0(Y, B)$. In C^*-algebraic deformation quantization, where $X = I$, both the situation that the field is trivial at $(0, 1]$ and the case that all fiber algebras $A_\hbar$ are pairwise non-isomorphic occur! The former happens, for example, in Weyl–Moyal quantization and its generalizations, whereas the latter occurs for certain noncommutative tori.

In any case, we see that a continuous field of C^*-algebras over the interval I is nothing but a $C(I)$ C^*-algebra A with an additional continuity property. Hence, in analogy with the notion of a generalized star-product introduced above, we may reformulate Def. II.1.2.5 in [**32**] as follows.

DEFINITION 7. *A strict quantization of a Poisson manifold P is a $C(I)$ C^*-algebra A such that*

(1) $A_0 = A/C(I,0)A \cong C_0(P)$ *as* C^**-algebras;*
(2) *There exists a cross-section* Q *of* π_0*, defined on a suitable Poisson subalgebra of* $C_0(P)$*, such that, in terms of* $Q_\hbar = \pi_\hbar \circ Q$*,*

$$\lim_{\hbar \to 0} \| \frac{i}{\hbar} (Q_\hbar(f)Q_\hbar(g) - Q_\hbar(g)Q_\hbar(f)) - Q_\hbar(\{f,g\}) \| = 0;$$

(3) *For each* $A \in A$*, the function* $\hbar \mapsto \|\pi_\hbar(A)\|$ *from* I *to* $\mathbb{R}^+$ *is continuous.*

Here the norm and the product are taken in $A_\hbar$.

The naive quantization of $A^*(G)$ by $C^*(G)$ can be amplified into a strict quantization [**32, 33, 37, 49**]. The $C(I)$ C^*-algebra quantizing $A^*(G)$ turns out to be $C^*(G_T)$, where G_T is the generalized tangent groupoid of G [**24**].

7. The category RKK

We now define categories C*(I) and RKK, which are the appropriate substitutes of C* and KK, respectively, if one works with $C(I)$ C^*-algebras rather than merely with C^*-algebras. Our goal being a quantization functor with codomain RKK, for pedagogical reasons we first introduce C*(I).

We first generalize a definition of Blanchard [**4**], who considered the case $B = C(X)$ (also cf. [**39**]).

DEFINITION 8. *Let* A *and* B *be* $C(X)$ C^**-algebras. An* A*-*B $C(X)$ *Hilbert bimodule is an* A*-*B *Hilbert bimodule for which the* A *action is* $C(X)$*-linear.*

The $C(X)$-linearity means the following: since the left action of A on E and the right action of B on E are both nondegenerate, they extend to the respective multiplier algebras, so that a priori one obtains two different actions of $C(X)$ on E, coming from A and B seen as $C(X)$ C^*-algebras. These actions must coincide. Consequently, one obtains a field $(E_x)_{x \in X}$ of A_x-B_x Hilbert bimodules, where

$$E_x = E \hat{\otimes}_B B_x$$

is the interior tensor product of E (as an A-B Hilbert bimodule) and B_x (as a B-B_x Hilbert bimodule). The left action of B on B_x is defined through $\pi_x : B \to B_x$ and left multiplication, the right action of B_x on itself is given by right multiplication, and the B_x-valued inner product on B_x is $\langle A, B \rangle = A^*B$ as usual. The left action of A_x on E_x is well defined because of the $C(X)$-linearity of the given A action on E. Thus one may think of an A-B $C(X)$ Hilbert bimodule as a field of A_x-B_x Hilbert bimodules with certain continuity properties following from the above definition; the special case $B = C(X)$ (so that $B_x = \mathbb{C}$) considered in [**4**] corresponds to a field of nondegenerate representations of A_x on a field of Hilbert spaces over X.

DEFINITION 9. *Let* X *be a compact Hausdorff space. The objects of the category* C*(X) *are* $C(X)$ C^**-algebras. The arrows are unitary isomorphism classes of* $C(X)$ *Hilbert bimodules. Matched arrows are composed through Rieffel's interior tensor product. The identity arrow* 1_A *at an object* A *is the class of* A*, seen as the canonical* A*-*A *Hilbert bimodule.*

When X is a point, one recovers the category C*. When $X = I$, one has the category C*(I). The categories C*(X) are the appropriate C^*-algebraic analogues of the categories k- Alg in the purely algebraic setting, where k is a commutative ring (cf. [**35**], Sect. 2.1); the objects of k- Alg are associative unital algebras over k, and the arrows are isomorphism classes of bimodules, composed using the obvious

tensor product. In particular, $\mathsf{C}^*(\mathrm{I})$ is the C^*-algebraic counterpart of the category $\mathbb{C}[[\hbar]]$- Alg. It can be shown that two $C(X)$ C^*-algebras are isomorphic as objects in $\mathsf{C}^*(\mathrm{X})$ iff they are Morita equivalent as $C(X)$ C^*-algebras (this means that they are Morita equivalent through an imprimitivity bimodule [**51**] that is also a $C(X)$ Hilbert bimodule). The purely algebraic counterpart of this result is that two objects are isomorphic in k- Alg iff they are Morita equivalent (in the usual algebraic sense); see Prop. 2.4 in [**35**], and also cf. [**6**].

We now define the category $\mathsf{RKK}(\mathrm{I})$, which refines $\mathsf{C}^*(\mathrm{I})$ in the same way that KK refines C^*. In fact, one may define a category $\mathsf{RKK}(\mathrm{X})$ for any compact Hausdorff space X [**48**]. As with KK, one starts with the notion of a Kasparov cycle. Given separable $C(X)$ C^*-algebras A and B, the elements of $\mathcal{RE}(X;A,B)$ are those elements $(E,F) \in \mathbb{E}(A,B)$ for which E is an A-B $C(X)$ Hilbert bimodule. There is no additional condition on F. The group $\mathcal{RKK}(X;A,B)$ is then defined in precisely the same way as $KK(A,B)$, as the quotient of $\mathcal{RE}(X;A,B)$ by the equivalence relation generated by unitary equivalence, translation of F along norm-continuous path, and the addition of degenerate Kasparov cycles. There is an intersection product

$$\mathcal{RKK}(X;A,B) \times \mathcal{RKK}(X;B,C) \to \mathcal{RKK}(X;A,C),$$

enabling one to define a category $\mathsf{RKK}(\mathrm{X})$ in the obvious way; the objects are $C(X)$ C^*-algebras, the arrows are the groups $\mathcal{RKK}(X;-,-)$, composed through the intersection product. We are now in a position to state

CONJECTURE 1. *There is a functor from* Poisson *to* $\mathsf{RKK}(I)$ *that on objects defines a strict quantization.*

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KORTEWEG-DE VRIES INSTITUTE FOR MATHEMATICS, UNIVERSITY OF AMSTERDAM, PLANTAGE MUIDERGRACHT 24, 1018 TV AMSTERDAM,THE NETHERLANDS
E-mail address: npl@science.uva.nl

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Star exponential functions for quadratic forms and polar elements

Hideki Omori, Yoshiaki Maeda, Naoya Miyazaki, and Akira Yoshioka

ABSTRACT. To describe star exponential functions for quadratic forms of the Weyl algebra W^2 rigorously, we introduce a complex 3-manifold Σ^3 and a family $\mathcal{M}_{\Sigma^3}$ of singular $\mathbb{C}_*$-bundle over Σ^3 with the singular $\mathbb{C}_*/\pm$ fiber, where the exponential map is understood as a map from the set of all quadratic forms of W^2 to $\mathcal{M}_{\Sigma^3}$. We also study the characteristic elements, called *polar elements*, which sit at the boundary of an open chart for $\mathcal{M}_{\Sigma^3}$, and show that they are essentially unique.

1. Introduction

Let W^2 be the Weyl algebra over $\mathbb{C}$ generated by $\hbar$, u and v with the relation:

$$[u, v]_* = -i\hbar, \tag{1.1}$$

where $[u, v]_* = u * v - v * u$ and $*$ denotes the product on W^2. Thinking of $\hbar$ as the Planck constant, we treat $\hbar$ as a positive parameter. Let $\mathcal{Q}$ be the set of all quadratic forms $Q_* = Q_*(a, b, c)$ in W^2 of the form

$$Q_* = au * u + b(u * v + v * u) + cv * v, \tag{1.2}$$

where $a, b, c \in \mathbb{C}$. The set $\mathcal{Q}$ is identified with $\mathbb{C}^3$ via the identification $(a, b, c) \longleftrightarrow Q_*(a, b, c)$.

The purpose of this paper is to establish the notion of the star exponential functions $\exp_* Q_*(a, b, c)$ via the Moyal product and the normal product, by studying evolution equations corresponding to these expressions (see [**9**]). However, we have to glue together these solutions in some manner. To obtain this, we develop a notion of "manifold" with an ambiguous property, which we call a *blurred manifold.*

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We introduce this notion via a toy model: Let $S^2 = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Given $\alpha, \delta \in S^2$, we set $\gamma = \gamma_{(\alpha,\delta)}$ to be a slit of S^2 joining from α to δ. Let S^α_δ, $\alpha \neq \delta$ be the set of all slits joining from α to δ in S^2.

Let $\mathbb{C}_* = \mathbb{C} - \{0\}$. To make a singular $\mathbb{C}_*$-bundle $\mathcal{L}$ over S^2, set $\mathbb{C}_\delta = \mathbb{C} - \{\delta\}$, and glue together $\mathbb{C} \times \mathbb{C}_*$ and $\mathbb{C} \times \mathbb{C}_*$ by the following *holomorphic diffeomorphism*:

$$T_\gamma : \mathbb{C}_\delta \times \mathbb{C}_* \to \mathbb{C}_\alpha \times \mathbb{C}_*, \quad T_\gamma(z,w) = \left(\frac{\alpha z + (1+\alpha\delta)}{z-\delta}, \frac{1}{\sqrt[(\gamma)]{z-\delta}} w \right). \tag{1.3}$$

for $z \in \mathbb{C}_\delta$. Here, the determination of the sign of the of the square root is fixed by the slit $\gamma = \gamma_{(\alpha,\delta)}$ and we denote it by $\sqrt[(\gamma)]{\ }$. The inverse of T_γ is given by

$$T_\gamma^{-1}(z,w) = \left(\frac{\delta z + (1+\alpha\delta)}{z-\alpha}, \frac{1}{\sqrt[(\gamma)]{z-\alpha}} w \right)$$

for $z \in \mathbb{C}_\alpha$.

Associated with the slit $\gamma_{(\alpha,\delta)}$ and the above coordinate transformation T_γ, we obtain a singular $\mathbb{C}_*$-bundle over S^2, denoted by $(\mathcal{L}, \gamma_{(\alpha,\delta)})$ such that

(L.1) The restriction $\mathcal{L}|_{S^2 - \gamma_{(\alpha,\delta)}}$ is a smooth $\mathbb{C}_*$-bundle over $S^2 - \gamma_{(\alpha,\delta)}$ with the transition function T_γ, $\gamma = \gamma_{(\alpha,\delta)}$.

(L.2) $\mathcal{L}|_{S^2 - \gamma_{(\alpha,\delta)}}$ is a $\mathbb{C}_*/\{\pm\}$-bundle over $\gamma_{(\alpha,\delta)}$.

We denote by $\mathcal{M}^{(\alpha,\delta)}_{S^2}$ the set of all singular $\mathbb{C}_*$-bundles $(\mathcal{L}, \gamma_{(\alpha,\delta)})$ over S^2, and set $\mathcal{M}_{S^2} = \cup_{\alpha,\delta \in S^2} \mathcal{M}^{(\alpha,\delta)}_{S^2}$.

We call $\mathcal{M}_{S^2}$ a "blurred manifold" over S^2. If we forget the $\pm$ ambiguity of $\mathcal{L}$, then by identifying $\pm w$ with w^2, $\mathcal{L}/\pm$ is naturally diffeomorphic to $\mathbb{C}^2 - \{0\}$, the tautological bundle over S^2. In this sense $\mathcal{M}_{S^2}$ is viewed as a "blurred double cover" of $\mathbb{C}^2 - \{0\}$.

We next introduce a complex manifold, denoted by Σ^3, as follows (cf. §5). We fix $\hbar$ as a nonnegative real number, and set

$$f_M(a,b,c) = 1 - 2ib\hbar - D\hbar^2, \quad f_N(a,b,c) = 1 + 2ib\hbar - D\hbar^2, \tag{1.4}$$

where $D = b^2 - ac$. We set

$$\begin{aligned} Z_M &= \mathbb{C}^3 - \{f_M = 0\}, \\ Z_N &= \mathbb{C}^3 - \{f_N = 0\}. \end{aligned}$$

We define a complex manifold Σ^3 given by the following transition function:

$$\begin{aligned} &T_\hbar : Z_M \subset \mathbb{C}^3 \to Z_N \subset \mathbb{C}^3, \\ &T_\hbar(a_M, b_M, c_M) = \frac{1}{f_M(a_M, b_M, c_M)} (a_M, b_M - iD_M\hbar, c_M). \end{aligned} \tag{1.5}$$

Let $f_{M,N} : \Sigma^3 \to S^2$ be a holomorphic map given by

$$f_{M,N}(a,b,c) = \begin{cases} f_M(a,b,c) & \text{for} \quad (a,b,c) \in \mathbb{C}^3 - \{f_N = 0\}, \\ f_N(a,b,c) & \text{for} \quad (a,b,c) \in \mathbb{C}^3 - \{f_N = 0\}. \end{cases} \tag{1.6}$$

We consider the pull-back of $\mathcal{M}^{(0,\infty)}_{S^2}$ via the map $f_{M,N}$, which is denoted by $\mathcal{M}_{\Sigma^3} = f^*_{M,N} \mathcal{M}^{(0,\infty)}_{S^2}$, that is, $\mathcal{M}_{\Sigma^3}$ contains the pull-backs of all singular $\mathbb{C}_*$-bundles $(f^*_{M,N}\mathcal{L}, f^{-1}_{M,N}\gamma_{(0,\infty)})$, where $\gamma \in S^0_\infty$.

In §3, we see that $\mathcal{M}_{\Sigma^3}$ allows us to intertwine the star exponential functions of quadratic functions on $\mathbf{C}^2$ via the Moyal product and the normal product. The slit $\gamma(0,\infty)$ is used to fix the sign of the square root appearing in the intertwiners.

Using $\mathcal{M}_{\Sigma^3}$, we define the map $\exp_* : \mathcal{Q} \to \mathcal{M}_{\Sigma^3}$ which rigorously gives the exponential map $\exp_* Q_*$ for any $Q_* \in \mathcal{Q}$ in §5.

The second purpose of this paper is to analyze the set $\mathcal{B}$ of the singular points of exponential functions $\exp_* tQ_*$ $(t \in \mathbb{C})$ via the Moyal product. It is given by

$$\mathcal{B} = \{\exp_* \frac{(2m+1)\pi}{2\hbar\sqrt{D}} Q_*(a,b,c) \mid m \in \mathbb{Z}\, D = b^2 - ac \neq 0\}.$$

We show that $\mathcal{B}$ consists of an essentially unique element (cf. Theorem 6.3).

Since $\mathcal{Q}$ is closed under the commutator bracket and forms a Lie algebra, we can define the *holomorphic involutive distribution* on $\mathcal{M}_{\Sigma^3}$ which gives the infinitesimal action of $\exp_* Q_*$. The *maximal integral submanifold* of this distribution gives a group-like object sitting in $\mathcal{M}_{\Sigma^3}$. The object seems to be a non-trivial double cover of $SL(2,\mathbb{C})$ and a local group, though it is *neither* a group nor a *point set.*

In relation to these objects, Olver [**5**] gives interesting examples of "non-associative local groups", which, however, seem different from ours. This problem will be studied in the forthcoming paper.

After submitting this manuscript, the referee pointed out an interesting paper by Burdet and Perrin [**2**] which gave a solution of evolution equations in the Weyl algebra and presented a twisted exponentials which played a role in the construction of the metaplectic representation. Our work seems to give a generalization of the metaplectic representation to the complex case. We thank the referee for his kind suggestions. These problems will be studied in the forthcoming paper.

2. The Weyl algebra and deformation quantizations

2.1. The formal Weyl algebra. Let W^2 be the Weyl algebra given by (1.1). It is known that W^2 is a noncommutative, associative algebra. Given $w \in W^2$, we have a unique expression for w as

$$w = \sum_{r=0} w_r \hbar^r, \qquad w_r = \sum_{r,mn} w_{r,nm} u^n * v^m \quad (w_{r,nm} \in \mathbb{C}). \tag{2.1}$$

Then $W^2 = \oplus_{r\geq 0} W_r^2$, where W_r^2 is homogeneous of degree r in $\hbar$.

According to the degree, we take the completion of W^2, which is denoted by $\tilde{W}^2$. Thus, $\tilde{W}^2$ contains formal power series of the form $\tilde{w} = \sum_{r=0}^{\infty} \tilde{w}_r \hbar^r$, where $\tilde{w}_r = \sum \tilde{w}_{r,nm} u^n * v^m$ (finite sum).

2.2. Deformation quantization of $\mathcal{P}(\mathbb{C}^2)$. Let us denote by $\mathcal{P}(\mathbb{C}^2)$ the set of all polynomials on $\mathbb{C}^2$. By a deformation quantization of $\mathcal{P}(\mathbb{C}^2)$ ([**1**], [**6**]), we mean an associative (noncommutative) product $*$ on $\mathcal{P}(\mathbb{C}^2)[[\hbar]]$ given as

$$f * g = \sum_{r=0}^{\infty} C_r(f,g)\hbar^r \quad \text{for } f,\ g \in \mathcal{P}(\mathbb{C}^2), \tag{2.2}$$

satisfying for every $r \geq 0$

(D1) $C_r(\cdot,\cdot)$ is a bi-differential operator.

(D2) $C_0(f,g) = f \cdot g,\ C_1(f,g) - C_1(g,f) = \partial_v f \cdot \partial_u g - \partial_u g \cdot \partial_v f.$

Among deformation quantizations of $\mathcal{P}(\mathbb{C}^2)$, we choose typical deformation quantizations of $\mathcal{P}(\mathbb{C}^2)$ which arise from the pseudo-differential operator calculus:

DEFINITION 2.1. Given $f, g \in \mathcal{P}(\mathbb{C}^2)$, we set the following:

$$f *_M g = \sum_{k=0}^{\infty} \frac{(i\hbar)^k}{2^k\, k!} f(\overleftarrow{\partial}_v \cdot \overrightarrow{\partial}_u - \overleftarrow{\partial}_u \cdot \overrightarrow{\partial}_v)^k g, \tag{2.3}$$

$$f *_N g = \sum_{k=0}^{\infty} \frac{(i\hbar)^k}{k!} f(\overleftarrow{\partial}_v \cdot \overrightarrow{\partial}_u)^k g, \tag{2.4}$$

where $(\overleftarrow{\partial}_v \cdot \overrightarrow{\partial}_u - \overleftarrow{\partial}_u \cdot \overrightarrow{\partial}_v)$ and $\overleftarrow{\partial}_u \cdot \overrightarrow{\partial}_v$ are bi-differential operators given by

$$f(\overleftarrow{\partial}_v \cdot \overrightarrow{\partial}_v)g = \partial_v f \cdot \partial_u g, \quad f(\overleftarrow{\partial}_v \cdot \overrightarrow{\partial}_u - \overleftarrow{\partial}_u \cdot \overrightarrow{\partial}_v)g = \partial_v f \cdot \partial_u g - \partial_u f \cdot \partial_v g. \tag{2.5}$$

The products $*_M$ and $*_N$ are called the Moyal product and the normal product, respectively.

We denote by α either of $*_M$ and $*_N$. Products on $\mathcal{P}(\mathbb{C}^2)$ can be regarded as realizations of W^2 as follows: Given an element $\tilde{w} \in \tilde{W}^2$ of the form $\tilde{w} = \sum \tilde{w}_r \hbar^r$, $\tilde{w}_r = \sum_{m,n} \tilde{w}_{r,mn} v^m * u^n$ $(\tilde{w}_{r,mn} \in \mathbb{C})$, we replace the product $*$ in W^2 by the product $*_\alpha$ for $\alpha = M, N$. Then, we have the element

$$\tilde{w}^\alpha = \sum \tilde{w}_r^\alpha \hbar^r, \qquad \tilde{w}_r^\alpha = \sum_{n,m} \tilde{w}_{r,nm} (u)^n_{*_\alpha} *_\alpha (v)^m_{*_\alpha}. \tag{2.6}$$

We denote by $\tilde{w}^\alpha$ the representation of $\tilde{w} \in W^2$ via $*_\alpha$.

2.3. Completion of $\mathcal{P}(\mathbb{C}^2)$. We take a completion of $\mathcal{P}(\mathbb{C}^2)$ by introducing the following semi-norm (cf. [3], [8]). Let $\mathcal{E}(\mathbb{C}^2)$ be the set of the entire functions on $\mathbb{C}^2$.

DEFINITION 2.2. Assume $p > 0$. We define the space $\mathcal{E}_p(\mathbb{C}^2)$ to consist of $f = f(u, v) \in \mathcal{E}(\mathbb{C}^2)$ satisfying

$$|f|_{p,\delta} = \sup_{(u,v)\in\mathbb{C}^2} |f(u,v)| e^{-\delta(|u|^2+|v|^2)^{p/2}} < \infty \quad \text{for all} \quad \delta > 0. \tag{2.7}$$

The family $\{|\cdot|_{p,\delta}\}_{\delta>0}$ gives a system of semi-norms on $\mathcal{E}_p(\mathbb{C}^2)$ defining a Fréchet space. Multiplication of functions is continuous, and then $\mathcal{E}_p(\mathbb{C}^2)$ is a Fréchet commutative algebra with respect to the canonical commutative multiplication.

The following result for the case of $*_\alpha = *_M$ is shown in [6]. A similar computation yields the case $*_\alpha = *_N$, which is easier than the case $*_\alpha = *_M$.

THEOREM 2.3. *Let $*_\alpha$ be the either of $*_N$ and $*_M$.*

(1) $(\mathcal{E}_p(\mathbb{C}^2), *_\alpha)$ for $0 < p \leq 2$ is a Fréchet noncommutative algebra.

(2) Assume $q > 0$ and $p > 2$ such that $\frac{1}{q} + \frac{1}{p} \geq 1$. Then, $\mathcal{E}_p(\mathbb{C}^2)$ is an $\mathcal{E}_q(\mathbb{C}^2)$ left- and right- module with respect to the product $*_\alpha$.

It is known that $\mathcal{E}_p(\mathbb{C}^2)$ $(p > 2)$ is not closed under the product $*_\alpha$ in general (cf. [8]). We define a Fréchet space by $\mathcal{E}_{2+}(\mathbf{C}^2) = \bigcap_{p>2} \mathcal{E}_p(\mathbb{C}^2)$. We focus the set of exponential functions of quadratic functions, a subset of $\mathcal{E}_{2+}(\mathbf{C}^2)$. Its product structure is not so obvious.

3. Extension of the equivalence of the deformation quantizations

It is known that any two deformation quantizations of $\mathcal{P}(\mathbb{C}^2)$ are mutually equivalent. We extend the equivalence between the Moyal product and the normal product on $\mathcal{P}(\mathbb{C}^2)[[\hbar]]$ to a set of exponential functions of quadratic functions on $\mathbb{C}^2$.

3.1. The formal equivalence between the Moyal and the normal product. Let $(\mathcal{P}(\mathbb{C}^2)[[\hbar]], *_M)$ and $(\mathcal{P}(\mathbb{C}^2)[[\hbar]], *_N)$ be the deformation quantizations of $\mathcal{P}(\mathbb{C}^2)[[\hbar]]$ given by (2.3) and (2.4), respectively. We define a formal differential operator T by

$$T = \exp\left(-\tfrac{\hbar}{2i}\partial_u\partial_v\right) = \sum_{k=0}^{\infty} \tfrac{1}{k!}\left(-\tfrac{\hbar}{2i}\right)^k (\partial_u\partial_v)^k. \tag{3.1}$$

We recall the following fact:

PROPOSITION 3.1. The formal differential operator T given by (3.1) yields the equivalence of $(\mathcal{P}(\mathbb{C}^2)[[\hbar]], *_M)$ and $(\mathcal{P}(\mathbb{C}^2)[[\hbar]], *_N)$. Namely, we have

$$T(f *_M g) = T(f) *_N T(g) \tag{3.2}$$

for every $f, g \in \mathcal{P}(\mathbb{C}^2)[[\hbar]]$.

PROOF. For given $F_M, F_N \in \mathcal{P}(\mathbb{R}^2)[[\hbar]]$, we define a Weyl-type pseudo-differential operator $W(F_M)$ and the standard pseudo-differential operator $P(F_N)$ acting on $C_0^\infty(\mathbb{R})$ by

$$(W(F_M)\phi)(u) = \int F_M(\tfrac{u+u'}{2}, v') e^{\frac{i}{2\hbar}(u-u')v'} \phi(u') du' dv', \tag{3.3}$$

and

$$(P(F_N)\phi)(u) = \int F_N(u, v') e^{\frac{i}{\hbar}(u-u')v'} \phi(u') du' dv', \tag{3.4}$$

respectively. It is known that there is a prescription from the Weyl-type pseudo-differential operator $W(F_M)$ to the standard pseudo-differential operator $P(F_N)$, given by

$$F_N = e^{-\frac{\hbar}{2i}\partial_u\partial_v} F_M \tag{3.5}$$

(cf [**4**]). Note that $W(F_M *_M G_M) = W(F_M)W(G_M)$ and $P(F_N *_N G_N) = P(F_N)P(G_N)$ for any $F_M, G_M \in \mathcal{P}(\mathbb{R})[[\hbar]]$, where we consider the Moyal product $*_M$ in (2.3) and the normal product $*_N$ in (2.4) by using the real coordinates u, v on $\mathbb{R}^2$. Thus, we have (3.2) for real polynomials $F, G \in \mathcal{P}(\mathbb{R}^2)[[\hbar]]$. Extending this relation to the complex 2-space $\mathbb{C}^2$, we have the desired result. □

Proposition 3.1 implies that the equivalence T given by (3.1) makes sense only for formal functions in $\hbar$.

3.2. Differential equations for extensions of T. We now extend T to a certain class of analytic functions in $\hbar$. For this purpose, we regard $\hbar$ as a non-negative real parameter.

We consider the following differential equation:

$$\begin{cases} \partial_t F = -\frac{1}{2i}\partial_u\partial_v F, \\ F(0,u,v) = g_M \exp(a_M u^2 + 2b_M uv + c_M v^2). \end{cases} \tag{3.6}$$

for $(a_M, b_M, c_M; g_M) \in \mathbb{C}^4$. We look for the analytic solution of (3.6) in t, which is unique if it exists. We assume that F has the following form:

$$F(t) = g_N(t)\exp(a_N(t)u^2 + 2b_N(t)uv + c_N(t)v^2). \tag{3.7}$$

Plugging (3.7) into (3.6), we get

$$\begin{cases} \frac{d}{dt}a_N(t) = 2ia_N(t)b_N(t), \\ \frac{d}{dt}b_N(t) = i(b_N(t)^2 + a_N(t)c_N(t)), \\ \frac{d}{dt}c_N(t) = 2ic_N(t)b_N(t), \\ \frac{d}{dt}g_N(t) = ib_N(t)g_N(t), \end{cases} \tag{3.8}$$

with the initial condition $(a_N(0), b_N(0), c_N(t); g_N(0)) = (a_M,\ b_M,\ c_M;\ g_M)$.

We find the solution: Let $\gamma = \gamma(0,\infty)$ be a slit on S^2. We set $\tilde{\gamma} = \hat{f}_M^{-1}(\gamma)$, $\hat{f}_M(t,a,b,c) = 1 - 2ibt - Dt^2$, $D = b^2 - ac$. To fix the sign of the square root of $\hat{f}_M(t,a,b,c)$, we fix the sign of $\sqrt{z}$ in S^2 with respect to γ. We denote it by $\sqrt[(\gamma)]{\hat{f}_M(t,a,b,c)}$. By a direct computation, we have

PROPOSITION 3.2. Fix a slit $\gamma(0,\infty) \in S^0_\infty$. Then, the solution of (3.8) is:

$$\begin{aligned} a_N(t) &= \frac{a_M}{1 - 2ib_M t - D_M t^2}, \quad b_N(t) = \frac{b_M - iD_M t}{1 - 2ib_M t - D_M t^2}, \\ c_N(t) &= \frac{c_M}{1 - 2ib_M t - D_M t^2}, \quad g_N(t) = \frac{g_M}{(1 - 2ib_M t - D_M t^2)^{1/2}}. \end{aligned} \tag{3.9}$$

where

$$(1 - 2ib_M t - D_M t^2)^{1/2} = \sqrt[(\gamma)]{\hat{f}_M(t, a_M, b_M, c_M)}$$

.

REMARK 3.3. The differential equation (3.8) for the case $a_M c_M = 0$ gives

$$\begin{aligned} a_N(t) &= \frac{a_M}{1 - ib_M t}, \ b_N(t) = \frac{b_M}{1 - ib_M t}, \\ c_N(t) &= \frac{c_M}{1 - ib_M t}, \ g_N(t) = \frac{g_M}{(1 - ib_M t)^{1/2}}. \end{aligned} \tag{3.10}$$

These are obtained as $a_M \to 0$ or $c_M \to 0$ in the solution (3.9). Hence (3.9) covers the case $a_M c_M = 0$.

We take a slit $\gamma \in S^0_\infty$. Replacing t by $\hbar$, we define the map

$$\begin{cases} T_\hbar(a_M, b_M, c_M) = (a_N(\hbar), b_N(\hbar), c_N(\hbar)) \\ \tilde{T}^{(\gamma)}_\hbar(a_M, b_M, c_M; g_M) = (a_N(\hbar), b_N(\hbar), c_N(\hbar); g_N(\hbar)) \end{cases} \tag{3.11}$$

where $(a_N(\hbar), b_N(\hbar), c_N(\hbar); g_N(\hbar))$ is given by (3.9). We also set the functions

$$f_M(a,b,c) = 1 - 2ib\hbar - D\hbar^2, \tag{3.12}$$

$$f_N(a,b,c) = 1 + 2ib\hbar - D\hbar^2. \tag{3.13}$$

and domains in $\mathbb{C}^3$ by

$$\begin{cases} Z_M = \mathbb{C}^3 - \{f_M = 0\}, \\ Z_N = \mathbb{C}^3 - \{f_N = 0\}, \\ \tilde{Z}_M = \{(a,b,c;g) \in \mathbb{C}^4 | (a,b,c) \in Z_M\}, \\ \tilde{Z}_N = \{(a,b,c;g) \in \mathbb{C}^4 | (a,b,c) \in Z_N\}. \end{cases} \tag{3.14}$$

PROPOSITION 3.4. For every $\gamma \in S^0_\infty$, the maps $T_\hbar : Z_M \to Z_N$ and $\tilde{T}^{(\gamma)}_\hbar : \tilde{Z}_M \to \tilde{Z}_N$ given by (3.11) are holomorphic diffeomorphisms for any fixed $\hbar$.

PROOF. Note that

$$\begin{cases} 1 + 2ib_N(\hbar)\hbar = \dfrac{(b_M^2 - a_M c_M)\hbar}{1 - 2ib_M\hbar - D_M\hbar^2}, \\ b_N(\hbar)^2 - a_N(\hbar)c_N(\hbar) = \dfrac{D_M}{1 - 2ib_M\hbar - D_M\hbar^2}. \end{cases} \tag{3.15}$$

We have

$$1 + 2ib_N(\hbar)\hbar - D_N(\hbar)\hbar^2 = \frac{1}{1 - 2ib_M\hbar - D_M\hbar^2}. \tag{3.16}$$

By using (3.16), the inverse $\tilde{T}^{(\gamma)\,-1}_\hbar(a_N, b_N, c_N; g_N) = (a_M, b_M, c_M; g_M)$ is given by

$$\begin{aligned} a_M &= \frac{a_N}{1 + 2ib_N\hbar - D_N\hbar^2}, \quad b_M = \frac{b_N + iD_N\hbar}{1 + 2ib_N\hbar - D_N\hbar^2}, \\ c_M &= \frac{c_N}{1 + 2ib_N\hbar - D_N\hbar^2}, \quad g_M = \frac{g_N}{(1 + 2ib_N\hbar - D_N\hbar^2)^{1/2}}. \end{aligned} \tag{3.17}$$

where $D_N = b_N^2 - a_N c_N$. Here, $(1 + 2ib_N\hbar - D_N\hbar^2)^{1/2}$ is fixed as in (3.9) after replacing $\hat{f}_M$ by $\hat{f}_N(t,a,b,c) = 1 + 2ibt - Dt^2$. We check that the inverse of $T_\hbar$ is obtained by the backward equation of (3.6):

$$\begin{cases} \partial_t \widehat{F} &= \frac{1}{2i}\partial_u \partial_v \widehat{F}, \\ \widehat{F}(0) &= q(a_N, b_N, c_N; g_N). \end{cases} \tag{3.18}$$

As in (3.9), we assume $\widehat{F}$ has the following form:

$$\widehat{F} = q(a_M(t), b_M(t), c_M(t); g_M(t)) \tag{3.19}$$

with

$$a_M(0) = a_N,\ b_M(0) = b_N,\ c_M(0) = c_N,\ g_M(0) = g_N. \tag{3.20}$$

□

DEFINITION 3.5. We call $\tilde{T}^{(\gamma)}_\hbar$ the *extension of* the equivalence T associated with the slit γ.

4. Star exponential functions for quadratic forms

We focus on a notion of transcendental elements for the Weyl algebra W^2. In the following, we attempt to describe the structure of the set of star exponential functions for the Weyl algebra W^2.

4.1. Star exponential functions of W^2. Given $w \in W^2$, its star exponential function is given formally by

$$exp_* tw = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\overbrace{w * \cdots * w}^{k}) \qquad t \in \mathbf{C} \tag{4.1}$$

which one can think of as a transcendental element for W^2. To define $exp_* tw$ for $w \in W^2$, we replace (4.1) by a solution of the following evolution equation;

$$\frac{\partial F}{\partial t} = w * F, \quad F(0) = 1. \tag{4.2}$$

We define the star exponential function $exp_* tw$ by $F(t)$ if (4.2) has a unique solution in t. We now attempt to handle the set of star exponential functions for quadratic forms in W^2.

Let Q_* be a quadratic form in W^2 given by

$$Q_* = au * u + b(u * v + v * u) + cv * v, \tag{4.3}$$

where $a, b, c \in \mathbf{C}$. We regard Q_* as a function of a, b, c and denote (4.3) by $Q_*(a, b, c)$.

We can consider Q_* as a function on $\mathbf{C}^2$ via either the Moyal product or the normal product:

$$Q_*^M = au^2 + 2buv + cv^2, \tag{4.4}$$

$$Q_*^N = au^2 + 2buv + cv^2 + ib\hbar. \tag{4.5}$$

We set

$$\tilde{\mathcal{M}}_0^{(2)} = \{ge^Q | \ Q = au^2 + 2buv + cv^2, g, a, b, c \in \mathbf{C}\}. \tag{4.6}$$

Then, $\tilde{\mathcal{M}}_0^{(2)}$ is closed subset of the Fréchet space $\mathcal{E}_{2+}(\mathbf{C}^2)$ parameterized by $\mathbf{C}^4$.

4.2. Star exponential function of quadratic forms via $*_\alpha$. Let Q_* be a quadratic form given by (4.3). As in the section 2.1, Q_* is realized as the quadratic function Q_*^M of (4.4) via the Moyal product. Replacing the evolution equation (4.2) via the Moyal product, we consider the equation:

$$\partial_t F = (Q_*^M) *_M F, \quad F(0) = 1. \tag{4.7}$$

Namely, we have $F(t) = e^{-i\hbar bt} F'(t)$. By (2.3), the evolution equation (4.7) becomes:

$$\begin{cases} \partial_t F &= Q_*^M F + \hbar i((cv + ba)\partial_u F - (au + bv)\partial_v F) \\ & \quad -\frac{\hbar^2}{4}(c\partial_u^2 F - 2b\partial_u \partial_v F + a\partial_v^2 F), \\ F(0) &= 1. \end{cases} \tag{4.8}$$

The solution $F(t)$ depends on the parameters a, b, c and we denote the solution by $F(t; a, b, c)$ if we wish to indicate the parameter dependence on a, b, c.

We define an open set of $\mathbf{C}^4$ as

$$\hat{U}_M = \{(t, a, b, c) \in \mathbf{C}^4 \,|\, \cos(\hbar\sqrt{D}\, t) \neq 0\} \tag{4.9}$$

where $D = b^2 - ac$.

PROPOSITION 4.1. There exists a unique analytic solution $F(t;a,b,c) \in \tilde{\mathcal{M}}_0^{(2)}$ of (4.8) on $\hat{U}_M$, given explicitly by

$$F(t,u,v) = \frac{1}{\cos(\hbar\sqrt{D}t)} \exp\left(\frac{Q}{\hbar\sqrt{D}} \tan(\hbar\sqrt{D}t)\right). \tag{4.10}$$

PROOF. We assume that the solution $F(t)$ of (4.8) has the following form:

$$F(t) = G_M(t)\exp(A_M(t)u^2 + 2B_M(t)uv + C_M(t)v^2) \tag{4.11}$$

Plugging (4.11) into (4.8), we have the following equations:

$$\begin{cases} \dot{A}_M = a + 2i\hbar(bA_M - aB_M) - \hbar^2(cA_M^2 - 2bA_MB_M + aB_M^2), \\ \dot{B}_M = b + 2i\hbar(cA_M - aC_M) \\ \qquad\qquad - \hbar^2(cA_MB_M - bA_MC_M - bB_M^2 + aB_MC_M), \\ \dot{C}_M = c + 2i\hbar(cB_M - bC_M) - \hbar^2(cB_M^2 - 2bB_MC_M + aC_M^2), \\ \dot{G}_M = -\frac{\hbar^2}{2}(cA_M - 2bB_M + aC_M)G_M, \end{cases} \tag{4.12}$$

with the initial conditions $G_M(0) = 1$, $A_M(0)=B_M(0)=C_M(0)=0$. Putting $A_M = ak(t)$, $B_M = bkt(t)$, $C_M = ck(t)$, we have

$$k' = 1 + \hbar^2 Dk^2. \tag{4.13}$$

Then, we have the solutions:

$$\begin{cases} A_M(t) = \dfrac{a}{\hbar\sqrt{D}} \tan(\hbar\sqrt{D}t), \\ B_M(t) = \dfrac{b}{\hbar\sqrt{D}} \tan(\hbar\sqrt{D}t), \\ C_M(t) = \dfrac{c}{\hbar\sqrt{D}} \tan(\hbar\sqrt{D}t), \\ G_M(t) = \dfrac{1}{\cos(\hbar\sqrt{D}t)}. \end{cases} \tag{4.14}$$

Thus, we have an unique analytic solution $F(t)$ of (4.8) in the form (4.11), where $(A_M(t), B_M(t), C_M(t); G_M(t))$ are given by (4.14). □

The appeared ambiguity in choice of the sign for the square root $\sqrt{D}$ in the solution (4.14) is resolved by the equation in (4.14). It is easily seen that the solution of (4.14) is still valid for the case of $D = 0$.

We now proceed as above for the case of the normal product $*_N$. Noting also that the quadratic form $Q_* \in W^2$ is also realized as the quadratic function Q_*^N of (4.5) via the normal product, we have the evolution equation:

$$\partial_t F = (Q_*^N) *_N F, \quad F(0) = 1 \tag{4.15}$$

where

$$Q_*^N = au^2 + 2buv + cv^2 + ib\hbar, \quad a,b,c \in \mathbf{C}. \tag{4.16}$$

By (2.4), the equation (4.15) becomes

$$\begin{cases} \partial_t F &= Q_*^N F + i\hbar(2bu + 2cv)\partial_u F - \hbar^2 c\partial_u^2 F \\ F(0) &= 1 \end{cases} \tag{4.17}$$

Define an open set by

$$\hat{U}_N = \{(t,a,b,c) \in \mathbf{C}^4 |\ ib\sin(2\hbar\sqrt{D}\,t) - \sqrt{D}\cos(2\hbar\sqrt{D}\,t) \neq 0\}$$

where $D = b^2 - ac$.

To write down the solution of (4.14), we have to fix the square root of the function $\sqrt{D}\cos(2\hbar\sqrt{D}\,t) - ib\sin(2\hbar\sqrt{D}\,t)$.

PROPOSITION 4.2. There exists a unique analytic solution $F(t) \in \tilde{\mathcal{M}}_0^{(2)}$ of (4.17) on $\hat{U}_N$ of the following form

$$F(t) = G_N(t)\exp(A_N(t)u^2 + 2B_N(t)uv + C_N(t)v^2), \tag{4.18}$$

where

$$\begin{cases} A_N(t) = \dfrac{a}{2\hbar}\dfrac{\sin(2\hbar\sqrt{D}\,t)}{(\sqrt{D}\cos(2\hbar\sqrt{D}\,t) - ib\sin(2\hbar\sqrt{D}\,t)}) \\ B_N(t) = \dfrac{i}{2\hbar}(1 - \dfrac{\sqrt{D}}{\sqrt{D}\cos(2\hbar\sqrt{D}\,t) - ib\sin(2\hbar\sqrt{D}\,t)}) \\ C_N(t) = \dfrac{c}{2\hbar}\dfrac{\sin(2\hbar\sqrt{D}\,t)}{(\sqrt{D}\cos(2\hbar\sqrt{D}\,t) - ib\sin(2\hbar\sqrt{D}\,t)}, \\ G_N(t) = \dfrac{D^{1/4}}{(\sqrt{D}\cos(2\hbar\sqrt{D}\,t) - ib\sin(2\hbar\sqrt{D}\,t))^{1/2}}, \end{cases} \tag{4.19}$$

$D = b^2 - ac$, and $G_N(t)$ has an ambiguity for the sign of the square root on its right hand side.

PROOF. We remark that if there exists a solution of (4.17) analytic in t, then it is unique. We look for a solution of (4.17) of the following form:

$$F(t) = G_N(t)\exp\left(A_N(t)u^2 + 2B_N(t)uv + C_N(t)v^2\right), \tag{4.20}$$

for some functions $A_N(t)$, $B_N(t)$, $C_N(t)$, $G_N(t)$ of t. Plugging (4.20) into (4.17), and comparing the coefficients in u^2, uv and v^2, we get

$$\begin{cases} \dot{A}_N = a + 4i\hbar bA_N - 4\hbar^2 cA_N^2, \\ \dot{B}_N = b + 2i\hbar(bB_N + cA_N) - 4\hbar^2 cA_N B_N, \\ \dot{C}_N = c + 4i\hbar cB_N - 4\hbar^2 cB_N^2, \\ \dot{G}_N = -2\hbar^2 cA_N g + ib\hbar G_N, \end{cases} \tag{4.21}$$

where $\dot{A}_N(t) = \frac{d}{dt}A_N(t)$ etc. and with the initial conditions

$$A_N(0) = B_N(0) = C_N(0) = 0\ G_N(0) = 1. \tag{4.22}$$

A_N is directly solvable in (4.21). Solving B_N, C_N, G_N successively, we have (4.19).

The solutions $A_N(t), B_N(t), C_N(t)$ and $G_N(t)$ are analytic under the assumptions of Proposition 4.2. The ambiguity in the sign for $\sqrt{D}$ in the solution (4.19) cancels out as before. However, the ambiguity for the sign of the square root of $\sqrt{D}\cos(2\hbar\sqrt{D}t) - ib\sin(2\hbar\sqrt{D}t)$ still remains. □

Recall the extension $\tilde{T}_\hbar^{(\gamma)}$ of T in the Definition 3.5. Let $(A_M(t), B_M(t), C_M(t); G_M(t))$ and $(A_N(t), B_N(t), C_N(t); G_N(t))$ be given by (4.10) and (4.19).

Then, we have

PROPOSITION 4.3. For a fixed slit $\gamma \in S^0_\infty$ and for the solution $(A_N(t), B_N(t), C_N(t); G_N(t))$ of (4.19), if the solution $(A_M(t), B_M(t), C_M(t); G_M(t)) \in Z_M \cap U_M$ of (4.14) and $(A_N(t), B_N(t), C_N(t); G_N(t)) \in Z_N \cap U_N$, then

$$\begin{cases} \tilde{T}^{(\gamma)}_\hbar(A_M(t), B_M(t), C_M(t); G_M(t)) = (A_N(t), B_N(t), C_N(t); G_N(t)), \\ \tilde{T}^{(\gamma)\ -1}_\hbar(A_N(t), B_N(t), C_N(t); G_N(t)) = (A_M(t), B_M(t), C_M(t); G_M(t)). \end{cases} \tag{4.23}$$

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5. Blurred manifold and exponential functions

We finally describe the exponential functions of quadratic forms in W^2.

5.1. Blurred manifold. To define the exponential function, we first summarize the tools briefly mentioned before. Let $S^2 = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and $\mathcal{M}^{(0,\infty)}_{S^2}$ the blurred manifold in the introduction, that is, the collection of $(\mathcal{L}, \gamma(0,\infty))$, where $\mathcal{L}$ is a singular $\mathbb{C}_*$-bundle over S^2, and $\gamma(0,\infty)$ is a slit satisfying (L1) and (L2).

We introduce a complex manifold denoted by Σ^3, as follows. We fix a nonpositive real number $\hbar$. We set

$$f_M(a,b,c) = 1 - 2ib\hbar - D\hbar^2, \quad f_N(a,b,c) = 1 + 2ib\hbar - D\hbar^2, \tag{5.1}$$

where $D = b^2 - ac$. We set

$$Z_M = \mathbf{C}^3 - \{f_M = 0\} \tag{5.2}$$

$$Z_N = \mathbf{C}^3 - \{f_N = 0\} \tag{5.3}$$

As stated in Introduction, we define a complex 3-manifold Σ^3 given by the following coordinate transformation: $T_\hbar : Z_M \to Z_N$:

$$T_\hbar(a_M, b_M, c_M) = \frac{1}{f_M(a_M, b_M, c_M)}(a_M, b_M - iD_M\hbar, c_M). \tag{5.4}$$

The inverse map of $T_\hbar$ is given by

$$T^{-1}_\hbar(a_N, b_N, c_N) = \frac{1}{f_M(a_N, b_N, c_N)}(a_N, b_N + iD_N\hbar, c_N).$$

Let $f_{M,N} : \Sigma^3 \to S^2$ be a holomorphic map given by

$$f_{M,N}(a,b,c) = \begin{cases} f_M(a,b,c) & \text{for} \quad (a,b,c) \in \mathbb{C}^3 - \{f_N = 0\} \\ f_N(a,b,c) & \text{for} \quad (a,b,c) \in \mathbb{C}^3 - \{f_N = 0\}. \end{cases} \tag{5.5}$$

We consider the pull-back $\mathcal{M}^{(0,\infty)}_{S^2}$ via the map $f_{M,N}$, which is denoted by $\mathcal{M}_{\Sigma^3} = f^*_{M,N}\mathcal{M}^{(0,\infty)}_{S^2}$, that is, $\mathcal{M}_{\Sigma^3}$ contains the pull-backs of all singular $\mathbb{C}_*$-bundles $(\mathcal{L}, \gamma_{(0,\infty)})$ with the slit $\gamma_{(0,\infty)}$ via $f_{M,N}$. We denote the element of $\mathcal{M}_{\Sigma^3}$ by $(f^*_{M,N}\mathcal{L}, \tilde{\gamma})$ where $\tilde{\gamma} = f^{-1}_{M,N}(\gamma(0,\infty))$.

5.2. Exponential functions. Let $\mathcal{Q}$ be the set of quadratic forms $Q_*{=}Q_*(a,b,c)$ given by (1.2). The space $\mathcal{Q}$ is identified with $\mathbb{C}^3$. In the following, we define the map

$$\exp_* : \mathbb{C}^3 \to \mathcal{M}_{\Sigma^3} \tag{5.6}$$

which rigorously define the exponential functions of the element Q_* of $\mathcal{Q}$. Set

$$U_M = \{(a_M, b_M, c_M) \in \mathbb{C}^3 \,|\, \cos(\hbar\sqrt{D_M} \neq 0\} \tag{5.7}$$

$$U_N = \{(a_N, b_N, c_N) \in \mathbf{C}^3 | ib\sin(2\hbar\sqrt{D_N}\, t) - \sqrt{D_N}\cos(2\hbar\sqrt{D_N}\, t) \neq 0\} \tag{5.8}$$

where $D_M = b_M^2 - a_M c_M, D_N = b_N^2 - a_N c_N$. We define the map

$$\exp_* : \mathbb{C}^3 \to f_{MN}^* \mathcal{M}_{S^2} \tag{5.9}$$

in terms of the local coordinates U_M and U_N: For fixed $\gamma \in S_\infty^0$, we set

$$\exp_{*_M}^{(\gamma)}(a_M, b_M, c_M) = (A_M(\hbar), B_M(\hbar), C_M(\hbar); G_M(\hbar)) \tag{5.10}$$

and

$$\exp_{*_N}^{(\gamma)}(a_N, b_N, c_N) = (A_N(\hbar), B_N(\hbar), C_N(\hbar); G_N(\hbar)) \tag{5.11}$$

where $(A_M(\hbar), B_M(\hbar), C_M(\hbar); G_M(\hbar))$ and $(A_N(\hbar), B_N(\hbar), C_N(\hbar); G_N(\hbar))$ are given by (4.14) and (4.19), respectively. Here, we note that the sign of the square root for $G_N(\hbar)$ is fixed so that Proposition 4.3 is valid for $t = \hbar$.

The maps given by (5.10) and (5.11) are obtained by solving the evolution equations which the star exponential function for quadratic functions via the Moyal product and the normal product satisfy (cf. §4). We see that (5.10) and (5.11) are identified with the transition function $\tilde{T}_\hbar$ given by (3.11).

This yields a rigorous map $\exp_* : \mathbb{C}^3 \to f_{M,N}^* \mathcal{M}_{S^2}$. In particular, for a quadratic form Q_* given by (1.2), $\exp_* Q_*$ is an element of $f_{M,N}^* \mathcal{M}_{S^2}$, which is expressed as follows: For any slit $\gamma \in S_\infty^0$, we consider a family of $\exp_*^{(\gamma)} Q_*$ labeled by γ. Then, $\exp_*^{(\gamma)} Q_*$ is expressed in terms of local coordinate neighborhoods U_M and U_N by

$$\exp_*^{(\gamma)} Q(a,b,c) = \begin{cases} \exp_{*_M}^{(\gamma)}(a,b,c) & \text{for } (a,b,c) \in U_M \\ \exp_{*_N}^{(\gamma)}(a,b,c) & \text{for } (a,b,c) \in U_N \end{cases} \tag{5.12}$$

We also denote by $\exp_{*_M}(a,b,c) = \exp_{*_M} Q_M^*$. (resp. $\exp_{*_N}(a,b,c) = \exp_{*_N} Q_N^*$). From (5.11) and (5.10), we can set $\exp_* tQ_*^M$ by replacing (a,b,c) with (ta, tb, tc).

6. Polar elements

We study singular elements in the star exponential functions $\exp_* Q_*$.

6.1. The standard polar element. By the definition of the open set U_M given by (5.7), we have

$$\partial U_M = \{(a,b,c) \in \mathbb{C}^3 \,|\, \sqrt{D} = \frac{\pi}{2\hbar}(2m+1),\ m \in \mathbb{Z}\}. \tag{6.1}$$

Let us choose the element $Q_*^0 \in \mathcal{Q}$ such that

$$Q_*^0 = u * v + v * u. \tag{6.2}$$

Let $\exp_* tQ_*^0$ be the star exponential function of Q_*^0, which is an element of $\mathcal{M}_{\Sigma^3}$. Let $(Q_*^0)^M$ and $(Q_*^0)^N$ be their representations via the Moyal product and the normal product. Namely, we have

$$(Q_*^0)^M = 2uv, \quad (Q_*^0)^N = uv + i\hbar. \tag{6.3}$$

We consider the star exponential functions of (6.3), which are the solutions of (4.7) and (4.15). We denote them by $\exp_{*_M} t(Q_*^0)^M$ and $\exp_{*_N} t(Q_*^0)^N$, respectively.

By looking at the explicit form of $\exp_{*_M} t(Q_*^0)^M$, we have that $\exp_{*_M} t(Q_*^0)^M$ has singularities at $t = \frac{\pi}{2\hbar}(2m+1), m \in \mathbb{Z}$.

DEFINITION 6.1. We call $\exp_* \frac{\pi}{2\hbar}(Q_*^0)$ the *standard polar element* and denote it by $\pm\epsilon_{00}$.

The expression for the standard polar element $\pm\epsilon_{00}$ is singular in the coordinate neighborhood U_M. However, by taking a slit $\gamma \in S_\infty^0$, $\pm\epsilon_{00}$ is expressed on U_N by

$$\pm\epsilon_{00}^{(\gamma)N} = \sqrt[(\gamma)]{-1} \exp\left(\tfrac{i}{\hbar} uv\right). \tag{6.4}$$

where $\pm\epsilon_{00}^{(\gamma)}$ is the representation of the standard polar element $\pm\epsilon_{00}$ with respect to the slit $\gamma \in S_\infty^0$.

6.2. Essential uniqueness of the polar element.

We set

$$\mathcal{B} = \left\{ \exp_* \tfrac{(2m+1)\pi}{2\hbar\sqrt{D}} Q_*(a,b,c) \,|\, a,b,c \in \mathbb{C},\ D = b^2 - ac \neq 0 \right\}. \tag{6.5}$$

DEFINITION 6.2. An element $\epsilon \in \mathcal{B}$ is called a *polar element.*

We now show that the polar element is essentially unique:

THEOREM 6.3. *$\mathcal{B}$ contains only the element $\pm\epsilon_{00}$: that is, for any slit $\gamma \in S_\infty^0$, $exp_*^{(\gamma)} \frac{(2m+1)\pi}{2\hbar\sqrt{D}} Q_*$ coincides with $\pm\epsilon_{00}^{(\gamma)}$ which is the representation of $\pm\epsilon_{00}$ with respect to the slit γ.*

PROOF. Take a slit $\gamma \in S_\infty^0$. Evaluating (4.19) at $t = \frac{(2m+1)\pi}{2\hbar\sqrt{D}}$, we have

$$\begin{aligned} &A_N\left(\tfrac{(2m+1)\pi}{2\hbar\sqrt{D}}\right) = C_N\left(\tfrac{(2m+1)\pi}{2\hbar\sqrt{D}}\right) = 0, \\ &B_N\left(\tfrac{(2m+1)\pi}{2\hbar\sqrt{D}}\right) = \tfrac{i}{\hbar}, \quad G_N\left(\tfrac{(2m+1)\pi}{2\hbar\sqrt{D}}\right) = \sqrt[(\gamma)]{-1} \end{aligned} \tag{6.6}$$

Thus, we have Theorem 6.3. □

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Department of Mathematics, Science University of Tokyo, Noda, Chiba, 278-8510, Japan
E-mail address: omori@ma.noda.sut.ac.jp

Department of Mathematics, Faculty of Science and Technology, Keio University, Hiyoshi, Yokohama, 223-8825, Japan
E-mail address: maeda@math.keio.ac.jp

Department of Mathematics, Faculty of Economics, Keio University, Hiyoshi, Yokohama, 223-8521, Japan
E-mail address: miyazaki@math.hc.keio.ac.jp

Department of Mathematics, Science University of Tokyo, Kagurazaka, Tokyo, 102-8601, Japan
E-mail address: yoshioka@rs.kagu.sut.ac.jp

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On traces for differential star products on symplectic manifolds

John Rawnsley

ABSTRACT. I describe a simple approach to the existence and uniqueness of traces for differential star products on symplectic manifolds based on Karabegov's local normalisation condition using ν-Euler derivations.

1. Introduction

The notion of a star product as a deformation theoretic approach to quantisation was introduced by Bayen, Flato et al [**1**]. A (formal) *star product* on a Poisson manifold $(M, \{\ ,\ \})$ is an associative multiplication $*$ on the algebra of formal power series $N[\![\nu]\!]$ with coefficients in the algebra N of smooth functions on M (here $\nu = i\hbar$). This is simultaneously a deformation of the usual multiplication of functions

$$u * v = uv + O(\nu)$$

and the commutator gives a deformation of the Poisson bracket

$$u * v - v * u = \nu\{u, v\} + O(\nu^2).$$

When the Poisson bracket comes from a symplectic structure, such a star product was shown to exist by Čech cohomological methods by de Wilde and Lecomte [**6**], and by geometrical methods by Fedosov [**7**], although various special cases were known earlier. More recently, Kontsevich [**17**] has shown by homotopy algebraic methods that all Poisson manifolds have star products.

For the purposes of representation theory, it is natural to ask if the algebra $(N[\![\nu]\!], *)$ determined by a star product has a trace, that is: a linear functional τ such that $\tau(u * v) = \tau(v * u)$. If M is not compact, then it is necessary to replace N by the ideal of compactly support functions N_c, since basic examples show that such traces are usually given by integration. For symplectic manifolds a positive answer is given by

THEOREM 5.1. (Fedosov, Nest–Tsygan) *Any star product on a symplectic manifold (M, ω) has a trace which is unique up to multiplication by an element of*

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$\mathbb{C}[\nu^{-1}, \nu]]$. *Every trace is given by a smooth density* $\rho \in C^\infty(M)[\nu^{-1}, \nu]]$:

$$\tau(u) = \int_M u\rho \frac{\omega^n}{\nu^n n!}.$$

In the general Poisson case, there is no natural volume form available with which to build linear functionals on spaces of functions, so it is unknown which star products have traces. Felder and Shoikhet [**11**] have shown that under a suitable condition on the Poisson tensor, a star product constructed by Kontsevich's method does have a trace. The case of a linear Poisson structure (dual of a Lie algebra) and the BCH star product has been studied by Bieliavsky et al [**2**].

In this talk I shall restrict attention to the symplectic case, and show how a local normalisation condition due to Karabegov [**16**] allows a simple piecing together argument to be used to prove the above theorem. Details will appear in a joint paper with Simone Gutt [**15**].

Finally, whilst the elementary methods we use here give existence and uniqueness of traces, this leaves us far from being able to write the trace densities down. An approach to doing this for his geometrically constructed star products has been described by Boris Fedosov [**10**].

2. Star products

Let (M, ω) be a connected symplectic manifold and N the algebra of smooth functions on M. Since ω is non-degenerate, contraction with ω gives a linear isomorphism between vector fields on M and 1-forms. For any function u the *Hamiltonian vector field* X_u is then the vector field which corresponds with the exact 1-form du. Since ω is closed the *Poisson bracket*

$$\{u, v\} = X_u(v) = \omega(X_v, X_u)$$

is a Lie algebra bracket on N. As an associative algebra under pointwise multiplication, N is abelian, and the Poisson bracket acts as derivations of the associative product,

$$\{uv, w\} = \{u, w\}v + u\{v, w\}, \forall u, v, w \in N,$$

so N is a *Poisson algebra.*

A *star product* is simultaneously a deformation of both the associative algebra and Lie algebra structures in the sense that we have an associative algebra structure denoted by $*$ on the space of formal power series $N[\![\nu]\!]$ whose leading term is the original multiplication, and such that the first order term of the commutator is the Poisson bracket. Thus we have

$$u * v = uv + \sum_{r \geq 1} \nu^r C_r(u, v)$$

with

$$C_1(u, v) - C_1(v, u) = \{u, v\}.$$

Whilst this is the most basic form of the definition of a star product, various additional assumptions are usually imposed. The constant function 1 will remain a unit element provided

$$C_r(1, u) = C_r(u, 1) = 0, \qquad \forall r \geq 1.$$

The star product is said to be *differential* if the C_r are differential operators in each argument.

Two star products, $*$ and $*'$, are said to be *equivalent* if there is a $\mathbb{R}[\![\nu]\!]$-linear map $T = \mathrm{Id} + \sum_{r\geq 1} \nu^r T_r$ of $N[\![\nu]\!]$ such that

$$u *' v = T(T^{-1}(u) * T^{-1}(v)).$$

T will preserve unities if $T_r(1) = 0$ for $r \geq 1$. If $*$ and $*'$ are differential, then an equivalence T is necessarily a series of differential operators.

In this talk we only consider differential star products and equivalences, and equivalences will preserve the unit element $1 \in N$. Differential operators can be restricted to functions on any open subset U, and also to the ideal N_c of compactly supported functions on M. A differential star product on $N[\![\nu]\!]$ is thus determined by its restriction $N_c[\![\nu]\!]$, and a differential star product on $N_c[\![\nu]\!]$ has a unique extension to $N[\![\nu]\!]$.

The basic example of all this is $\mathbb{R}^{2n}$ with the standard constant symplectic structure $\Omega = \frac{1}{2}\sum_{ij} \Omega_{ij} dx^1 \wedge dx^j$. The *Moyal star product* $*_M$ is given by

$$u *_M v = uv + \sum_{r\geq 1} \nu^r C_r^M(u, v)$$

where

$$C_r^M(u,v) = \frac{1}{2^r r!} \sum_{\substack{i_1 \dots i_r \\ j_1 \dots j_r}} \Lambda^{i_1 j_1} \dots \Lambda^{i_r j_r} \frac{\partial^r u}{\partial x^{i_1} \dots \partial x^{i_r}} \frac{\partial^r v}{\partial x^{j_1} \dots \partial x^{j_r}}$$

and Λ^{ij} is the inverse of the matrix Ω_{ij}.

Locally, any symplectic manifold is isomorphic to $(\mathbb{R}^{2n}, \Omega)$ by choosing Darboux coordinates, and any star product on a contractible open set in $\mathbb{R}^{2n}$ is equivalent to the restriction of Moyal. See [**14**] for an account of star products which is based heavily on this local equivalence.

3. Traces

DEFINITION 3.1. Let $*$ be a star product on (M, ω) then a *trace* is a $\mathbb{C}[\![\nu]\!]$-linear map $\tau\colon N_c[\![\nu]\!] \to \mathbb{C}[\nu^{-1}, \nu]\!]$ satisfying

$$\tau(u * v) = \tau(v * u).$$

REMARK 3.2. Since any $\mathbb{C}[\nu^{-1}, \nu]$-multiple of a trace is clearly a trace, it is not actually necessary to work with Laurent series, but we do so for two reasons. Firstly, in the formula for the trace of a pseudo-differential operator a factor of ν^{-n} occurs, and secondly, the presence of such a factor simplifies equation (4.1) arising below.

If τ is a non-trivial trace it is determined by its restriction to N_c, and can then expand it as a formal power series of linear functionals $\tau_s\colon N_c \to \mathbb{R}$

$$\tau(u) = \nu^r \sum_{s\geq 0} \nu^s \tau_s(u)$$

where we assume $\tau_0 \neq 0$. The condition to be a trace takes the form

$$\tau_k(\{u, v\}) + \tau_{k-1}(C_2^-(u, v)) + \dots + \tau_0(C_{k+1}^-(u, v)) = 0 \tag{3.1}$$

for $k = 0, 1, 2, \dots$, where C_r^- denotes the antisymmetric part of C_r.

REMARK 3.3. In [**5**] the notion of a *closed star product* was introduced and related to cyclic cohomology of the algebra of functions. The existence of a closed star product was proved in [**19**]; see also [**3, 20**]. A star product is closed if

$$\int_M C_r^-(u,v)\omega^n = 0, \qquad \forall u, v \in N_c \tag{3.2}$$

for all $1 \leq r \leq n$ where $2n$ is the dimension of M. If (3.2) actually holds for all $r \geq 1$ then $u * v$ is said to be *strongly closed.* Thus to be strongly closed is the same as requiring that

$$\tau(u) = \int_M u\omega^n$$

be a trace on $N_c[\![\nu]\!]$.

For $k = 0$, equation (3.1) reduces to the condition $\tau_0(\{u, v\}) = 0$ for all u, v. This condition is solved by

LEMMA 3.4. *Let (M, ω) be a connected symplectic manifold. If $\sigma\colon N_c \to \mathbb{R}$ is a linear map with $\sigma(\{u, v\}) = 0$ for all $u, v \in N_c$ then $\sigma(u) = c\int_M u\omega^n$ for some constant c.*

See [**4**] or [**15**] for a proof.

Thus any trace has the form

$$\tau(u) = a\nu^r\left(\int_M u\omega^n + \sum_{k\geq 1}\nu^k\tau_k(u)\right)$$

where a is non-zero. We can divide by a and multiply by ν^{-n-r} to bring τ into the form

$$\int_M u\frac{\omega^n}{\nu^n} + \nu^{-n}\sum_{k\geq 1}\nu^k\tau_k(u)$$

which is still a trace. A trace of this form will be said to be *standard* c.f. [**13**].

As observed in [**4**], this is enough to show that any two traces τ and τ' for the same star product are proportional by an element of $\mathbb{C}[\nu^{-1}, \nu]\!]$. For, if τ is standard, the leading term of τ' is a multiple $c\nu^r$ of the integral, hence is equal to the leading term of τ multiplied by $c\nu^{r+n}$. But then $\tau' - c\nu^{r+n}\tau$ is a trace which vanishes to at least order $r + 1$ in ν. This argument can be repeated indefinitely to show that $\tau' = c\nu^{r+n}(1 + \sum_k \nu^k c_k)\tau$. Notice that, in particular, if τ and τ' are standard then $\tau' = (1 + \sum_k \nu^k c_k)\tau$. This proves

THEOREM 3.5. (Nest & Tsygan [**18**]) *If $*$ is a star product on a connected symplectic manifold (M, ω) then any two traces for $*$ are proportional by an element of $\mathbb{C}[\nu^{-1}, \nu]\!]$.*

4. The local case

In the case of $\mathbb{R}^{2n}$ with its standard constant 2-form Ω, then

$$\tau_M(u) = \int_{\mathbb{R}^{2n}} u\,\frac{\Omega^n}{\nu^n n!}$$

is a trace on compactly supported functions for the Moyal star product $*_M$. The Moyal star product and this trace have an important homogeneity property [**16**].

If we take a conformal vector field ξ on $\mathbb{R}^{2n}$, so $\mathcal{L}_\xi\Omega = \Omega$ then $D_M = \xi + \nu\frac{\partial}{\partial\nu}$ is a derivation of $*_M$ and τ_M satisfies

$$\tau_M(D_M u) = \nu\frac{\partial}{\partial\nu}\tau_M(u). \tag{4.1}$$

If $*$ is any star product defined on an open ball U in $\mathbb{R}^{2n}$ then it is equivalent to the restriction of the Moyal star product by a map $T = \mathrm{Id} + \sum_{k\geq 1}\nu^k T_k$ with

$$T(u * v) = T(u) *_M T(v)$$

and then we see that

$$\tau(u) = \int_U T(u)\frac{\Omega^n}{\nu^n}$$

is a trace for $*$. Each T_k is a differential operator, so it has a formal adjoint T_k' so that, if we put $T' = \mathrm{Id} + \sum_{k\geq 1}\nu^k T_k'$, then

$$\tau(u) = \int_U uT'(1)\frac{\Omega^n}{\nu^n}$$

for $u \in C_c^\infty(U)$. If we put $\rho = T'(1) \in C^\infty(U)[\nu^{-1}, \nu]\!]$ then

$$\tau(u) = \int_U u\rho\frac{\Omega^n}{\nu^n}.$$

If τ is standard then $\rho = 1 + \sum_{k\geq 1}\nu^k\rho_k$.

Further $D = T^{-1}\circ D_M \circ T$ will be a derivation of $*$ and satisfies the transform of equation (4.1):

$$\tau(Du) = \nu\frac{\partial}{\partial\nu}\tau(u).$$

D has the form $\xi + \nu\frac{\partial}{\partial\nu} + D'$. Local derivations of this form are used in proving existence of star products by Čech cohomology methods and in [**14**] they are given a special name:

DEFINITION 4.1. Let (M, ω) be a symplectic manifold. Say that a derivation D of $N[\![\nu]\!], *$ on an open set U is **ν-Euler** if it has the form

$$D = \nu\frac{\partial}{\partial\nu} + X + D' \tag{4.2}$$

where X is conformally symplectic ($\mathcal{L}_X\omega = \omega$) and $D' = \sum_{r\geq 1}\nu^r D_r'$ with the D_r' differential operators on U.

Note that on an open set where we have a conformally symplectic vector field X then the condition $\mathcal{L}_X\omega = \omega$ implies $\omega = d(i(X)\omega)$ so ω is exact. Thus conformally symplectic vector fields only exist locally in general, so we also cannot ask for global ν-Euler derivations. Two local ν-Euler derivations defined on the same open set will differ by a ν-linear derivation, and so the difference is $\nu^{-1}\times$ inner. This means that $\tau\circ D$ will be independent of D and thus is globally defined as an $\mathbb{R}$-linear functional, even if D is not.

DEFINITION 4.2. A standard trace τ is said to be *normalised* if it satisfies the analogue of the Moyal homogeneity condition:

$$\tau(Du) = \nu\frac{\partial}{\partial\nu}\tau(u)$$

on any open set where there are ν-Euler derivations and for any such local ν-Euler derivation D.

This condition was introduced by Karabegov [**16**].

It is clear that the pull-back of the Moyal trace by an equivalence with the Moyal star product is normalised, so a normalised trace always exists for any star product on an open ball in $\mathbb{R}^{2n}$.

PROPOSITION 4.3. (Karabegov [**16**]) *If τ and τ' are normalised traces for the same star product on an open ball U in $\mathbb{R}^{2n}$ then $\tau = \tau'$.*

PROOF. Neither trace can be zero, and they are proportional by Theorem 3.5, so there is a power series of constants $1 + c\nu^r + \dots$ with $\tau' = (1 + c\nu^r + \dots)\tau$. If $\tau' \neq \tau$ then there is a first $r > 0$ where $c \neq 0$. Then we substitute in the normalisation condition to give

$$\begin{aligned}(1 + c\nu^r + \dots)\tau(Du) &= \tau'(Du)\\ &= \nu\frac{\partial}{\partial\nu}\tau'(u)\\ &= \nu\frac{\partial}{\partial\nu}((1 + c\nu^r + \dots)\tau(u))\\ &= (rc\nu^r + \dots)\tau(u) + (1 + c\nu^r + \dots)\tau(Du)\end{aligned}$$

which implies that $c = 0$. This contradiction shows that $\tau' = \tau$. □

5. The global case

Let (M, ω) be a connected symplectic manifold. Then we can cover M by Darboux charts U which are diffeomorphic to open balls in $\mathbb{R}^{2n}$ and such that all non-empty intersections are also diffeomorphic to open balls. Let $*$ be a star product on M and then the restriction of $*$ to U has a normalised trace with density $\rho_U \in C^\infty(U)[\![\nu]\!]$. If we have two open sets U, V which overlap, then on the intersection both ρ_U and ρ_V will determine normalised traces on an open ball in $\mathbb{R}^{2n}$. Since there is only one such trace, $\rho_U = \rho_V$ on $U \cap V$. It follows that there is a globally defined function ρ on M such that $\rho_U = \rho|_U$. Set

$$\tau(u) = \int_M u\rho\frac{\omega^n}{\nu^n}$$

for $u \in C_c^\infty(M)$. It is easy to see that τ is a normalised trace.

Combining this with Theorem 3.5 and the fact that the normalised trace we just constructed has a smooth density we obtain:

THEOREM 5.1 (Fedosov [**8, 9**], Nest–Tsygan [**18**]). *On a connected symplectic manifold (M, ω) any differential star product has a unique normalised trace. Any trace is multiple of this and is given by a smooth density.*

REMARK 5.2. In [**18**] a proof that any trace has a smooth density is given using cyclic cohomology.

The defining condition for a trace can be viewed as saying that τ is invariant under inner automorphisms. We can now deduce that they are in fact automatically invariant under all automorphisms.

COROLLARY 5.3. *Any trace is invariant under all $\mathbb{C}[\![\nu]\!]$-linear automorphisms of the star product.*

PROOF. A smooth trace is a multiple (in $\mathbb{C}[\nu^{-1}, \nu]\!]$) of a normalised trace. The transform of a ν-Euler derivation by a $\mathbb{C}[\![\nu]\!]$-linear automorphism is again a ν-Euler derivation, thus the transform of a normalised trace by a $\mathbb{C}[\![\nu]\!]$-linear automorphism is again a normalised trace, and so is equal to the original normalised trace. □

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MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY CV4 7AL, UNITED KINGDOM

E-mail address: J.Rawnsley@warwick.ac.uk

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Quantum G-manifolds

Joseph Donin

ABSTRACT. Let G be a semisimple Lie group, $\mathfrak{g}$ its Lie algebra, and $U_h(\mathfrak{g})$ the corresponding quantum group. We consider some fundamental examples of $U_h(\mathfrak{g})$-invariant one and two parameter quantizations on G-manifolds.

1. Introduction

Passing from classical mechanics to quantum mechanics involves replacing the commutative function algebra, $\mathcal{A}$, of classical observables on the appropriate phase space, M, with a noncommutative (deformed) algebra, $\mathcal{A}_t$, of quantum observables (see [**BFFLS**] where the deformation quantization scheme is developed). The algebra $\mathcal{A}$ is a Poisson algebra and the product in $\mathcal{A}_t$ is given by a power series in the formal parameter t with leading term the original commutative product and with leading term in the commutator given by the Poisson bracket. If the classical system is invariant under a Lie group of symmetries, G, then G acts on the phase space (and thus on $\mathcal{A}$) and the associated quantum system often retains the group of symmetries. In particular, the algebra $\mathcal{A}_t$ is often invariant under the action of G, or under the action of its universal enveloping algebra $U(\mathfrak{g})$.

Modern field-theoretical models and, in particular, the problem of incorporating gravity into a quantum field theory led to the requirement of deforming (quantizing) the group symmetry and the phase space themselves. This is one of the reasons for the interest in quantum groups. The quantum group, $U_h(\mathfrak{g})$, defined by Drinfeld and Jimbo is a deformation of $U(\mathfrak{g})$ as a Hopf algebra. The quantization of the phase space and its symmetry group corresponds to a $U_h(\mathfrak{g})$ invariant deformation of the algebra $\mathcal{A}_t$, which leads us to the problem of two parameter (or double) quantization, $\mathcal{A}_{t,h}$, of the function algebra for a $U(\mathfrak{g})$ invariant Poisson structure. In other words, the problem of two parameter (or double) quantization appears if we want to quantize the Poisson bracket in such a way that multiplication in the quantized algebra is invariant under the quantum group action.

In the present talk we consider examples of one and two parameter invariant quantizations of Poisson function algebras on some natural G-manifolds.

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In fact, we will regard $\mathcal{A}$ as a sheaf of algebras and $\mathcal{A}_t$ ($\mathcal{A}_{t,h}$) as a deformation of that sheaf. We call a *quantum* G-manifold a manifold M endowed with the deformed sheaf of non-commutative algebras.

In Section 2 we review some facts on quantum groups related to our problem. Note that we only consider the case of semisimple G. In Section 3 we formulate precisely the problem and reduce it to the problem of a G-invariant non-associative quantization. Note that in our setting we suppose that the action of G on the space of functions on M does not deform. In Section 4 we describe Poisson brackets admissible for $U_h(\mathfrak{g})$-invariant one and two parameter quantizations.

Beginning from Section 5 we consider examples of $U_h(\mathfrak{g})$-invariant quantization on some G-manifolds. Namely, we consider two-sided and Ad-action G on itself, coadjoint action G on $\mathfrak{g}^*$, semisimple orbits in $\mathfrak{g}^*$, and quotients of G by reductive subgroups of maximal rank.

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2. Quantum groups

We will consider quantum groups in the sense of Drinfeld, [**Dr2**], as Hopf algebras being deformed universal enveloping algebras. So we will regard that $U_h(\mathfrak{g}) = U(\mathfrak{g})[[h]]$ as a topological $\mathbb{C}[[h]]$-module and $U_h(\mathfrak{g})/hU_h(\mathfrak{g}) = U(g)$ as a Hopf algebra over $\mathbb{C}$. In particular, the deformed comultiplication in $U_h(\mathfrak{g})$ has the form

$$\Delta_h = \Delta + h\Delta_1 + o(h), \tag{2.1}$$

where Δ is the comultiplication in the universal enveloping algebra $U(\mathfrak{g})$. One can prove, [**Dr2**], that the map $\Delta_1 : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is such that $\Delta_1 - \sigma\Delta_1 = \delta$ (σ is the usual permutation) being restricted to $\mathfrak{g}$ gives a map $\delta : \mathfrak{g} \to \wedge^2\mathfrak{g}$ which is a 1-cocycle and defines the structure of a Lie coalgebra on $\mathfrak{g}$ (the structure of a Lie algebra on the dual space $\mathfrak{g}^*$). The pair $(\mathfrak{g}, \delta)$ is called a quasiclassical limit of $U_h(\mathfrak{g})$.

In general, a pair $(\mathfrak{g}, \delta)$, where $\mathfrak{g}$ is a Lie algebra and δ is such a 1-cocycle, is called a Lie bialgebra. It is proven, [**EK**], that any Lie bialgebra $(\mathfrak{g}, \delta)$ can be quantized, i.e., there exists a quantum group $U_h(\mathfrak{g})$ such that the pair $(\mathfrak{g}, \delta)$ is its quasiclassical limit.

A Lie bialgebra $(\mathfrak{g}, \delta)$ is said to be a coboundary one if there exists an element $r \in \wedge^2\mathfrak{g}$, called the classical r-matrix, such that $\delta(x) = [r, \Delta(x)]$ for $x \in \mathfrak{g}$. Since δ defines a Lie coalgebra structure, r has to satisfy the so-called classical Yang-Baxter equation which can be written in the form

$$[\![r, r]\!] = \varphi, \tag{2.2}$$

where $[\![\cdot, \cdot]\!]$ stands for the Schouten bracket and $\varphi \in \wedge^3\mathfrak{g}$ is an invariant element. We denote the coboundary Lie bialgebra by $(\mathfrak{g}, r)$.

In case $\mathfrak{g}$ is a simple Lie algebra, the most known r-matrix is the Sklyanin-Drinfeld one:

$$r = \sum_{\alpha} X_\alpha \wedge X_{-\alpha}, \tag{2.3}$$

where the sum runs over all positive roots; the root vectors X_α are chosen in such a way that $(X_\alpha, X_{-\alpha}) = 1$ for the Killing form $(\cdot,\cdot)$. This is the only r-matrix of weight zero, [**SS**], and its quantization is the Drinfeld-Jimbo quantum group. A classification of all r-matrices for simple Lie algebras was given in [**BD**].

From results of Drinfeld, [**Dr1**], and of Etingof and Kazhdan, [**EK**], one can derive the following description of quantum groups associated with semisimple Lie algebras.

PROPOSITION 2.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then*

a) Any Lie bialgebra $(\mathfrak{g}, \delta)$ is a coboundary one.

b) The quantization, $U_h(\mathfrak{g}, r)$, of any coboundary Lie bialgebra $(\mathfrak{g}, r)$ exists and is equal to $U(\mathfrak{g})[[h]]$ as a topological $\mathbb{C}[[h]]$-algebra.

c) The comultiplication in $U_h(\mathfrak{g}, r)$ has the form

$$\Delta_h(x) = F_h^{-1}\Delta(x)F_h, \qquad x \in U(\mathfrak{g}), \tag{2.4}$$

where $F_h \in U(\mathfrak{g})^{\otimes 2}[[h]]$ and can be chosen in the form

$$F_h = 1 \otimes 1 + \frac{h}{2}r + o(h). \tag{2.5}$$

Moreover,

$$(\varepsilon \otimes 1)F_h = (1 \otimes \varepsilon)F_h = 1 \otimes 1, \tag{2.6}$$

where $\varepsilon : U_h(\mathfrak{g}, r) \to \mathbb{C}[[h]]$ is the counit in $U_h(\mathfrak{g}, r)$, which coincides with the natural extension of the counit $U(\mathfrak{g}) \to \mathbb{C}$.

d) Antipode in $U_h(\mathfrak{g}, r)$ has the form

$$S_h(x) = w^{-1}S(x)w, \qquad x \in U(\mathfrak{g}), \tag{2.7}$$

where $w = \sum_i (SF_{1i})F_{2i}$. Here S is antipode in $U(\mathfrak{g})$ defined as $x \mapsto -x$ for $x \in \mathfrak{g}$, and $\sum_i F_{1i} \otimes F_{2i} = F_h$.

It follows from the coassociativity Δ_h that F_h satisfies the equation

$$\Phi_h \cdot (\Delta \otimes id)(F_h) \cdot (F_h \otimes 1) = (id \otimes \Delta)(F_h) \cdot (1 \otimes F_h) \tag{2.8}$$

for some invariant element $\Phi_h \in U(\mathfrak{g})^{\otimes 3}[[h]]$.

It follows from (2.8) that if F_h has the form (2.5), then the coefficient by h in Φ_h vanishes. Moreover, the coefficient by h^2 is up to a factor the element φ from (2.2), i.e.,

$$\Phi_h = 1 \otimes 1 \otimes 1 + h^2\varphi + o(h^2). \tag{2.9}$$

In addition, it follows from (2.8) that Φ_h satisfies the pentagon identity

$$\begin{aligned}(id^{\otimes 2} \otimes \Delta)(\Phi_h) \cdot (\Delta \otimes id^{\otimes 2})(\Phi_h) = \\ (1 \otimes \Phi_h) \cdot (id \otimes \Delta \otimes id)(\Phi_h) \cdot (\Phi_h \otimes 1).\end{aligned} \tag{2.10}$$

REMARK 2.2. One can suppose that an invariant element Φ_h of the form (2.9) satisfying (2.10) is given in advance. Such Φ_h can be constructed independently, [**Dr2**]. Then, for any element r obeying (2.2) one can find F_h of the form (2.5) such that (2.8) holds. So, we may take such F_h to define comultiplication (2.4) in the quantum group $U_h(\mathfrak{g}, r)$. Moreover, one can construct Φ_h satisfying the additional relation, [**DS2**]:

$$\Phi_h \Phi_h^S = 1, \tag{2.11}$$

where S is antipode in $U(\mathfrak{g})$ and $\Phi_h^S = (S \otimes S \otimes S)(\Phi_h)$. In the following we fix such a Φ_h.

The Hopf algebra $U = U_h(\mathfrak{g}, r)$ is quasitriangular with the universal R-matrix

$$\mathcal{R} = F_{h21}^{-1} e^{\frac{h}{2}\mathbf{t}} F_h = 1 \otimes 1 + h(\frac{\mathbf{t}}{2} + r) + \cdots, \tag{2.12}$$

where $\mathbf{t} \in \mathrm{Sym}^2 \mathfrak{g}$ is an invariant element. It is easy to check that $\mathcal{R}$ satisfies the relations, [**Dr2**]

$$\Delta_h'(x) = \mathcal{R}\Delta_h(x)\mathcal{R}^{-1}, \tag{2.13}$$

where Δ_h' denotes the opposite comultiplication to Δ_h, and

$$\begin{aligned} \Delta_{h1}\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23} \\ \Delta_{h2}\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{12}. \end{aligned} \tag{2.14}$$

Here subscripts indicate the position of the corresponding tensor factors. For example, if $a = a_1 \otimes a_2$, then $a_{13} = a_1 \otimes 1 \otimes a_3$ and $a_{21} = a_2 \otimes a_1$, Δ_{h1} denotes applying the comultiplication in $U_h(\mathfrak{g}, r)$ to the first factor.

3. $U_h(\mathfrak{g})$-invariant quantization

Let G be a semisimple connected complex Lie group with the Lie algebra $\mathfrak{g}$. Let M be a G-manifold and $\mathcal{A} = \mathcal{A}(M)$ a sheaf of functions on M. It may be the sheaf of analytic, smooth, or algebraic functions, dependingly of the type of M. Then $U(\mathfrak{g})$ acts on sections of $\mathcal{A}$, and the multiplication in $\mathcal{A}$ is $U(\mathfrak{g})$-invariant, i.e.,

$$bm(x \otimes y) = m\Delta(b)(x \otimes y) \qquad b \in U(\mathfrak{g}),\ x, y \in \mathcal{A}.$$

By a deformation quantization of $\mathcal{A}$ we mean a sheaf of associative algebras, $\mathcal{A}_h$, which is equal to $\mathcal{A}[[h]]$ as a $\mathbb{C}[[h]]$-module, with multiplication in $\mathcal{A}_h$ of the form

$$m_h = \sum_{k=0}^{\infty} h^k m_k,$$

where $m_0 = m$ is the usual commutative multiplication in $\mathcal{A}$ and m_k, $k > 0$, are bidifferential operators vanishing on constants. The action $U(\mathfrak{g})$ on $\mathcal{A}$ is naturally extended to the action of the $\mathbb{C}[[h]]$-algebra $U(\mathfrak{g})[[h]] = U_h(\mathfrak{g})$ on the $\mathbb{C}[[h]]$-module $\mathcal{A}[[h]] = \mathcal{A}_h$.

We will also consider two parameter quantizations on M. A two parameter quantization of $\mathcal{A}$ is an algebra $\mathcal{A}_{t,h}$ equal to $\mathcal{A}[[t, h]]$ as a $\mathbb{C}[[t, h]]$-module and having a multiplication of the form

$$m_{t,h} = m_0 + tm_1' + hm_1'' + o(t, h). \tag{3.1}$$

We are going to study quantizations of $\mathcal{A}$ which are invariant under the $U_h(\mathfrak{g})$-action, i.e., under the comultiplication Δ_h. The invariance means that

$$bm_h(x \otimes y) = m_h\Delta_h(b)(x \otimes y) \qquad b \in U(\mathfrak{g}),\ x, y \in \mathcal{A}, \tag{3.2}$$

where m_h is one or two parameter multiplication. Of course, for a two parameter $U_h(\mathfrak{g})$-invariant multiplication, $m_{t,h}$, the multiplication $m_t = m_{t,0}$ is $U(\mathfrak{g})$-invariant.

We call m_h satisfying (3.2) a $U_h(\mathfrak{g})$-*invariant quantization* of the G-manifold M.

DEFINITION 3.1. A $\mathbb{C}[[h]]$-linear map $\mu_h : \mathcal{A}_h \otimes \mathcal{A}_h \to \mathcal{A}_h$ is called a Φ_h-*associative* multiplication if

$$\mu_h(\Phi_1 x \otimes \mu_h(\Phi_2 y \otimes \Phi_3 z))) = \mu_h(\mu_h(x \otimes y) \otimes z) \quad \text{for } x, y, z \in \mathcal{A},$$

where $\Phi_h = \Phi_1 \otimes \Phi_2 \otimes \Phi_3$ (summation implicit).

We say that a Φ_h-associative multiplication $\mu_h = \sum_{k=0}^{\infty} h^k \mu_k$ gives a Φ_h-associative quantization of $\mathcal{A}$ if $\mu_0 = m_0$, the usual multiplication in $\mathcal{A}$, and μ_k, $k > 0$, are bidifferential operators vanishing on constants.

PROPOSITION 3.2. *There is a natural one-to-one correspondence between $U_h(\mathfrak{g})$-invariant and $U(\mathfrak{g})$-invariant Φ_h-associative quantizations of $\mathcal{A}$. Namely, if μ_h is a $U(\mathfrak{g})$-invariant Φ_h-associative multiplication in $\mathcal{A}[[h]]$, then*

$$m_h = \mu_h F_h \tag{3.3}$$

gives a $U_h(\mathfrak{g})$-invariant associative multiplication in $\mathcal{A}[[h]]$.

PROOF. This follows immediately from (2.4) and (2.8). This follows also from the categorical interpretation of Φ_h and F_h, [**Dr2**], [**DGS**]. □

This proposition shows that given a $U(\mathfrak{g})$-invariant Φ_h-associative quantization of $\mathcal{A}$, we obtain $U_h(\mathfrak{g}, r)$-invariant quantizations of $\mathcal{A}$ for all quantum groups $U_h(\mathfrak{g}, r)$ from Proposition 2.1 b), applying formula (3.3) with the appropriate F_h from (2.5).

4. Poisson brackets associated with $U_h(\mathfrak{g})$-invariant quantization

Let $\mathcal{A}$ be a function algebra on M. A bivector field on M we call a *bracket* on $\mathcal{A}$ if it induces a skew-symmetric map $f : \mathcal{A}^{\otimes 2} \to \mathcal{A}$.

For an element $\psi \in \wedge^k \mathfrak{g}$ we denote by ψ_M the k-vector field on M which is induced by the action map $\mathfrak{g} \to \mathrm{Vect}(M)$.

A bracket f is a Poisson one if the Schouten bracket $[\![f, f]\!]$ is equal to zero.

DEFINITION 4.1. A G-invariant bracket f on M we call a φ-*bracket* if

$$[\![f, f]\!] = -\varphi_M, \tag{4.1}$$

where $\varphi \in \wedge^3 \mathfrak{g}$ is an invariant element.

PROPOSITION 4.2. *Let $\mathcal{A}_h$ be a $U(\mathfrak{g})$-invariant Φ_h-associative quantization with multiplication $\mu_h = m_0 + h\mu_1 + o(h)$, where m_0 is the multiplication in $\mathcal{A}$. Then the map $f : \mathcal{A}^{\otimes 2} \to \mathcal{A}$, $f(a, b) = \mu_1(a, b) - \mu_1(b, a)$, is a φ-bracket for φ from (2.9).*

PROOF. A direct computation. □

In what follows, we denote $U_h(\mathfrak{g}) = U_h(\mathfrak{g}, -r)$.

COROLLARY 4.3. *Let $\mathcal{A}_h$ be a $U_h(\mathfrak{g})$-invariant associative quantization with multiplication $m_h = m_0 + hm_1 + o(h)$. Then the corresponding Poisson bracket $p(a, b) = m_1(a, b) - m_1(b, a)$ has the form*

$$p(a, b) = f(a, b) - r_M(a, b), \tag{4.2}$$

where r is the r-matrix corresponding to $U_h(\mathfrak{g})$ and f is a φ-bracket with $\varphi = [\![r, r]\!]$.

PROOF. Follows from Proposition 4.2 and (3.3). □

REMARK 4.4. Note that a Poisson bracket of the form (4.2) makes M into a Poisson G-manifold with respect to the action of Poisson-Lie group G endowed with the Poisson bracket $r^L - r^R$, where r^L (r^R) is the extension of r-marix r as a left- (right-) invariant bivector field on G.

Now, let us consider a two parameter $U_h(\mathfrak{g})$-invariant quantization (3.1). With such a quantization one associates two Poisson brackets: the bracket $s(a,b) = m_1'(a,b) - m_1'(b,a)$ along t, and the bracket $p(a,b) = m_1''(a,b) - m_1''(b,a)$ along h. It is easy to check that p and v are compatible Poisson brackets, i.e., their Schouten bracket $[\![p,s]\!]$ is equal to zero. So, we have

COROLLARY 4.5. *Let $\mathcal{A}_{t,h}$ be a $U_h(\mathfrak{g})$-invariant associative quantization of the form (3.1). Then the Poisson bracket $p(a,b) = m_1''(a,b) - m_1''(b,a)$ has the form*

$$p(a,b) = f(a,b) - r_M(a,b),$$

where r is the r-matrix corresponding to $U_h(\mathfrak{g})$ and f is a φ-bracket with $\varphi = [\![r,r]\!]$. The Poisson bracket $s(a,b) = m_1'(a,b) - m_1'(b,a)$ is $U(\mathfrak{g})$-invariant and compatible with f (and thus with p), i.e., f satisfies (4.1) and the additional condition

$$[\![f,s]\!] = 0. \tag{4.3}$$

PROOF. Similar to Corollary (4.3). □

So, the problem of quantizing on a G-manifold M is divided into two parts: 1) a classification of G-invariant bivector fields, f, on M satisfying (4.1) and (4.3) (if a G-invariant Poisson bracket s on M is given); 2) constructing one or two parameter G-invariant and Φ_h-associative quantizations of brackets (4.1) (and s).

5. Quantization on symmetric spaces and on G

5.1. Quantization on symmetric spaces. Let G be a semisimple connected Lie group. Let us recall that a symmetric space over G is a space of the form $M = G/H$, where H is a subgroup of G such that $G_0^\sigma \subset H \subset G^\sigma$, where G^σ is the set of fixed points of σ, an involutive automorphism of G, and G_0^σ is the identity component of G^σ. The automorphism σ induces an automorphism of the Lie algebra $\mathfrak{g}$ of G that we also denote by σ.

Let r be an r-matrix on $\mathfrak{g}$ and $U_h(\mathfrak{g},r)$ the corresponding quantum group. In [**DS1**], the following statement is proven.

THEOREM 5.1. *Let M be a symmetric space over a semisimple Lie group G, σ the corresponding involution of $\mathfrak{g}$. Let an r-matrix r on $\mathfrak{g}$ be such that the element $[\![r,r]\!] = \varphi$ is σ-invariant. Then there exists a $U_h(\mathfrak{g},r)$-invariant quantization on M.*

REMARK 5.2. Note that the σ-invariance of φ implies that $\varphi_M = 0$ and r_M is a Poisson bracket. So, in the case of symmetric space M the invariant part f of the corresponding Poisson bracket (4.2) may be taken to be zero.

5.2. Quantization of two-sided action of G on itself. Let G be as in the previous subsection. Consider G as a $G \times G$-manifold with the two-sided action of G on itself. More precisely, by the two-sided action we mean the action of $G \times G$ on G defined as follows: $(g_1,g_2) \triangleright g = g_1 g g_2^{-1}$, $(g_1,g_2) \in G \times G$, $g \in G$. Then, the group G may be considered as a symmetric space, $G = (G \times G)/H$, where H is the

diagonal. In this case σ on $\mathfrak{g}\oplus\mathfrak{g}$, the Lie algebra of $G\times G$, is the usual permutation: $\sigma(x,y)=(y,x)$.

Let r_1 and r_2 be two r-matrices on $\mathfrak{g}$ such that

$$[\![r_1,r_1]\!]=[\![r_2,r_2]\!]=\varphi. \tag{5.1}$$

Then element r_1+r_2 obviously defines a σ-invariant r-matrix on $\mathfrak{g}\oplus\mathfrak{g}$. The quantum group $U_h(\mathfrak{g},r_1)\otimes U_h(\mathfrak{g},r_2)$ may be considered as a quantization of the Lie bialgebra $\mathfrak{g}\oplus\mathfrak{g}$ with r-matrix r_1+r_2. Applying Theorem 5.1 we obtain

PROPOSITION 5.3. *Let G be a semisimple Lie group. Let r_1 and r_2 be r-matrices on its Lie algebra satisfying (5.1). Then there exists a $U_h(\mathfrak{g},r_1)\otimes U_h(\mathfrak{g},r_2)$-invariant quantization of the two-sided action of $G\times G$ on G.*

REMARK 5.4. It is clear that the presence of a $U_h(\mathfrak{g},r_1)\otimes U_h(\mathfrak{g},r_2)$-action on a $\mathbb{C}[[h]]$-module V is equivalent to the presence of two commuting homomorphisms: $\rho_1: U_h(\mathfrak{g},r_1)\to \mathrm{End}(V)$ and $\rho_2: U_h(\mathfrak{g},r_2)\to \mathrm{End}(V)$.

In the previous proposition ρ_1 and ρ_2 are induced by the maps

$$\rho_1(x)=x^R,\quad \rho_2(x)=(x^S)^L,\qquad x\in\mathfrak{g}, \tag{5.2}$$

where x^R (x^L) is a right- (left-) invariant vector field on G extending x and $x^S=S(x)=-x$. So, the previous proposition states the existence of an invariant quantization on G with respect to the both ρ_1- and ρ_2- actions simultaneously.

REMARK 5.5. The element Φ_h from Remark 2.2 associated to both $U_h(\mathfrak{g},r_1)$ and $U_h(\mathfrak{g},r_2)$ induces in the obvious way the corresponding element $\Phi_h\otimes\Phi_h$ associated to $U_h(\mathfrak{g},r_1)\otimes U_h(\mathfrak{g},r_2)$. It follows from (5.2) that the image of $\Phi_h\otimes\Phi_h$ by the $G\times G$-action on G is equal to $\Phi_h^R\Phi_h^{SL}$. It follows from invariance of Φ_h that $\Phi_h^R=\Phi_h^L$, so, applying (2.11), we see that $\Phi_h^R\Phi_h^{SL}=1\otimes 1\otimes 1$.

This implies that the usual multiplication in the function algebra on G is $\Phi_h\otimes\ \Phi_h$-associative. So, accordingly to Proposition (3.2), the quantization from the conclusion of Proposition 5.3 may be given explicitly as

$$m_h=mF_{h1}^R(F_{h2}^S)^L, \tag{5.3}$$

where F_{h1} and F_{h2} are elements from (2.4) associated to $U_h(\mathfrak{g},r_1)$ and $U_h(\mathfrak{g},r_2)$, respectively, and m is the usual multiplication.

REMARK 5.6. It is easy to calculate that, up to a factor, the Poisson bracket on G corresponding to the quantization from Proposition 5.3 has the form

$$r_1^R+r_2^L, \tag{5.4}$$

where r_i^L (r_i^R), $i=1,2$, is the extension of r_i as a left- (right-) invariant bivector field on G. In particular, if we take $r_1=r$, where r is the Sklyanin-Drinfeld r-matrix (2.3) and $r_2=-r$, then (5.4) becomes

$$r^R-r^L, \tag{5.5}$$

which is the Sklyanin-Drinfeld Poisson bracket on G.

REMARK 5.7. The following quantization on G generalizes a quantization given in [**Ta**]:

$$\widetilde{\mu}_h=mF_{h1}^R(F_{h2}^{-1})^L. \tag{5.6}$$

Consider the space of functions on G, $\mathcal{A}$, as a two-sided module over $U(\mathfrak{g})$. It is easy to check that the multiplication $\widetilde{\mu}_h$ makes $\mathcal{A}$ into an algebra invariant under the left action of $U_h(\mathfrak{g}, r_1)$ and the right action of $U_h(\mathfrak{g}, r_2)$.

This algebra may be considered as a left-invariant algebra over the quantum group $U_h(\mathfrak{g}, r_1) \otimes U_h^{cop}(\mathfrak{g}, r_2)$, where $U_h^{cop}(\mathfrak{g}, r_2)$ is $U_h(\mathfrak{g}, r_2)$ with the opposite comultiplication, in the following way:

$$(x, y) \triangleright a = x \triangleright a \triangleleft \bar{y}, \qquad x \in U_h(\mathfrak{g}, r_1),\ y \in U_h(\mathfrak{g}, r_2),\ a \in \mathcal{A},$$

where $\bar{y}$ means applying to y antipode from $U_h(\mathfrak{g}, r_2)$.

5.3. Quantization of *Ad*-action of G on itself. Consider the group G as a G-manifold with respect to the *Ad*-action: $g_1 \triangleright g = g_1 g g_1^{-1}$. The corresponding action $U(\mathfrak{g})$ on $\mathcal{A}(G)$ is obtained by the embedding $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ from the two-sided action $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ on $\mathcal{A}(G)$ considered above. In the quantum case, given a $U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$-invariant quantization, m_h, of $\mathcal{A}(G)$, the corresponding embedding $\Delta_h : U_h(\mathfrak{g}) \to U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ does not give a $U_h(\mathfrak{g})$-invariant quantization of the function algebra, because this embedding is not a morphism of coalgebras. Nevertheless, the multiplication m_h can be modified to give a quantization of the *Ad*-action.

Namely, the following lemma holds.

LEMMA 5.8. *Let $\mathcal{A}$ be an algebra with $U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$-invariant multiplication m_h. Let the action $U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ on $\mathcal{A}$ be given by two commuting actions ρ_1 and ρ_2 of $U_h(\mathfrak{g})$. Let the action $U_h(\mathfrak{g})$ on the $\mathbb{C}[[h]]$-module $\mathcal{A}$ be induced by the embedding $\Delta_h : U_h(\mathfrak{g}) \to U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$. Then the multiplication*

$$\mu_h'(a \otimes b) = \mu_h(\rho_2(\mathcal{R}_1)a \otimes \rho_1(\mathcal{R}_2)b) \tag{5.7}$$

is associative and $U_h(\mathfrak{g})$-invariant. Here $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2$ is the R-matrix of the quantum group $U_h(\mathfrak{g})$.

PROOF. A direct computation using relations (2.14). □

REMARK 5.9. Consider the quantum group $U_h(\mathfrak{g}) \otimes^{\mathcal{R}} U_h(\mathfrak{g})$ that is equal to $U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ as an algebra but with the following comultiplication $\widetilde{\Delta}_h$:

$$\widetilde{\Delta}_h(x \otimes y) = (x_1 \otimes \mathcal{R}_1 y_1 \overline{\mathcal{R}}_1) \otimes (\mathcal{R}_2 x_2 \overline{\mathcal{R}}_2 \otimes y_2),$$

where $x_1 \otimes x_2 = \Delta_h x$, $y_1 \otimes y_2 \Delta_h y$. Here $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2$ and $\mathcal{R}^{-1} = \overline{\mathcal{R}}_1 \otimes \overline{\mathcal{R}}_2$. It follows from (2.13) that the embedding $\Delta_h : U_h(\mathfrak{g}) \to U_h(\mathfrak{g}) \otimes^{\mathcal{R}} U_h(\mathfrak{g})$ is a morphism of bialgebras and the multiplication μ_h' constructed in Lemma 5.8 is $U_h(\mathfrak{g}) \otimes^{\mathcal{R}} U_h(\mathfrak{g})$-invariant.

The following lemma allows us to construct $U_h(\mathfrak{g})$-invariant quantizations from $U_h(\mathfrak{g})^{cop}$-invariant ones.

LEMMA 5.10. *Let $\mathcal{A}$ be an algebra with $U_h(\mathfrak{g})^{cop}$-invariant multiplication m_h, where $U_h(\mathfrak{g})^{cop}$ is $U_h(\mathfrak{g})$ with the opposite comultiplication. Then the multiplication*

$$\mu_h'(a \otimes b) = \mu(\mathcal{R}_1 a \otimes \mathcal{R}_2 b) \tag{5.8}$$

is $U_h(\mathfrak{g})$-invariant.

PROOF. A direct computation using relation 2.13. □

In particular, Lemmas 5.8, 5.10 may be used to obtain a $U_h(\mathfrak{g},r_1)\otimes^R U_h(\mathfrak{g},r_2)$-invariant quantization on G either from $U_h(\mathfrak{g},r_1)\otimes U_h(\mathfrak{g},r_2)$-invariant quantization (5.3) or from $U_h(\mathfrak{g},r_1)\otimes U_h^{cop}(\mathfrak{g},r_2)$-invariant quantization (5.6). So, we obtain

COROLLARY 5.11. *There exists a $U_h(\mathfrak{g})$-invariant quantization of the Ad-action G on G.*

REMARK 5.12. Using (5.3) and Lemma 5.8, it is easy to calculate that, up to a factor, the invariant part of the Poisson bracket corresponding to the quantization from the previous corollary is:

$$f(a,b)=(t_1^La)(t_2^Rb)-(t_1^Lb)(t_2^Ra), \tag{5.9}$$

where $\mathbf{t}=t_1\otimes t_2$ is the invariant element of $Sym^2\mathfrak{g}$. Here x^L (x^R) is the extension of $x\in\mathfrak{g}$ as a left- (right-) invariant vector field on G.

6. Quantization on $\mathfrak{g}^*$

Let $\mathfrak{g}$ be a complex Lie algebra. Then, the symmetric algebra $S\mathfrak{g}$ can be considered as a function (polynomial) algebra on $\mathfrak{g}^*$. The algebra $U(\mathfrak{g})$ is included in the family of algebras $(S\mathfrak{g})_t=T(\mathfrak{g})[t]/J_t$, where J_t is the ideal generated by the elements of the form $x\otimes y-\sigma(x\otimes y)-t[x,y]$, $x,y\in\mathfrak{g}$, σ is the permutation. By the PBW theorem, $(S\mathfrak{g})_t$ is a free module over $\mathbb{C}[t]$. We have $(S\mathfrak{g})_0=S\mathfrak{g}$, so this family of quadratic-linear algebras gives a $U(\mathfrak{g})$-invariant quantization of $S\mathfrak{g}$ by the Lie bracket s.

It turns out that for $\mathfrak{g}=sl(n)$ this picture can be extended to the quantum case, [**Do2**]. Namely,

THEOREM 6.1. *Let $\mathfrak{g}=sl(n)$. Then, there exist deformations, σ_h and $[\cdot,\cdot]_h$, of both the mappings σ and $[\cdot,\cdot]$ such that the two parameter family of algebras*

$$(S\mathfrak{g})_{t,h}=T(\mathfrak{g})[[h]][t]/J_{t,h},$$

where $J_{t,h}$ is the ideal generated by elements of the form $x\otimes y-\sigma_h(x\otimes y)-t[x,y]_h$, $x,y\in\mathfrak{g}$, gives a $U_h(\mathfrak{g})$-invariant quantization of the Poisson-Lie bracket $[\cdot,\cdot]$ on $\mathfrak{g}^*$.

Using Corollary 4.5, let us describe the Poisson brackets corresponding to $(S\mathfrak{g})_{t,h}$.

PROPOSITION 6.2. *The pair of brackets corresponding to the quantization $(S\mathfrak{g})_{t,h}$ consists of two compatible Poisson brackets:*

s (along t) is the Poisson-Lie bracket;

p (along h) is a quadratic Poisson bracket of the form $p=f-\{\cdot,\cdot\}_r$, where f is the invariant quadratic bracket which is defined by a unique up to a factor invariant map

$$f:\wedge^2\mathfrak{g}\to S^2\mathfrak{g}, \tag{6.1}$$

and $\{\cdot,\cdot\}_r$ is the r-matrix bracket. Moreover, $[\![s,f]\!]=0$ and $[\![f,f]\!]=-\overline{\varphi}$, where $\overline{\varphi}$ has the form $\overline{\varphi}(a,b,c)=[\varphi_1,a][\varphi_2,b][\varphi_3,c]$, and $\varphi=\varphi_1\wedge\varphi_2\wedge\varphi_3=[\![r,r]\!]$. Recall that φ is a unique up to a factor invariant element of $\wedge^3\mathfrak{g}$.

Note that the bracket f from the above proposition can be restricted to any orbit of $SL(n)$ in $sl(n)^*$. This follows from the following general statement, [**Do1**].

PROPOSITION 6.3. *Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$, $s = [\cdot,\cdot]$ the Poisson-Lie bracket on $\mathfrak{g}^*$. Let $f = \{\cdot,\cdot\}$ be an invariant bracket on an open set of $\mathfrak{g}^*$ such that the Schouten bracket $[\![s, f]\!]$ is a 3-vector field, ψ, which can be restricted to an orbit $\mathcal{O}$ of G. Then f can be restricted to $\mathcal{O}$.*

Note that an invariant bivector field defined on a part of an orbit is in fact defined on the whole orbit.

REMARK 6.4. Using quantized Verma modules, one can prove that the family $(S\mathfrak{g})_{t,h}$ can be restricted to any semisimple orbit to give a $U_h(\mathfrak{g})$-invariant quantization of the Kirillov bracket on it.

REMARK 6.5. One can prove, [**Do1**], that for a simple $\mathfrak{g} \neq sl(n)$ there are no φ-brackets on $\mathfrak{g}^*$ compatible with the Poisson-Lie bracket. Hence, in this case a two parameter $U_h(\mathfrak{g})$-invariant quantization on $\mathfrak{g}^*$ does not exist.

It seems that for a simple $\mathfrak{g} \neq sl(n)$ there exist no φ-brackets on $\mathfrak{g}^*$, which implies that on $\mathfrak{g}^*$ even a one parameter $U_h(\mathfrak{g})$-invariant quantization does not exist. This statement could be easily derived from Conjecture 9.1 at the end of the paper.

REMARK 6.6. Note that on a neighborhood of zero in $\mathfrak{g}$ a φ-bracket exists. Indeed, due to the map $\exp : \mathfrak{g} \to G$ the bracket (5.9) may be carried over from G to a neighborhood of zero in $\mathfrak{g}$.

Note also that the bracket of the form (5.9) is correctly defined on $gl(n)$ (considered as an associative algebra) and being restricted to $sl(n)$ gives a quadratic bracket which coincides with (6.1).

REMARK 6.7. It follows from the previous remark that for any simple $\mathfrak{g}$, a φ-bracket exists on any (not necessarily semisimple) orbit in $\mathfrak{g}^*$. Indeed, multiplying a given orbit by a constant one can suppose that it goes through the neighborhood of zero where the bracket (5.9) is defined. Now, the statement follows from Proposition 6.3.

Since bracket (6.1) is compatible with the Poisson-Lie bracket, it follows that for $\mathfrak{g} = sl(n)$ on any orbit in $\mathfrak{g}^*$ there exists a φ-bracket compatible with the Kirillov bracket.

As we will see in Subsection 7.2, in case $\mathfrak{g} \neq sl(n)$ not all orbits (even semisimple) admit φ-brackets compatible with the Kirillov brackets on them.

7. Quantization on semisimple orbits in $\mathfrak{g}^*$

Let G be a complex connected simple Lie group with the Lie algebra $\mathfrak{g}$. Let $U_h(\mathfrak{g}) = U_h(\mathfrak{g}, -r)$ be the corresponding quantum group.

Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$, the Levi factor of a parabolic subalgebra. Let L be a Lie subgroup of G with Lie algebra $\mathfrak{l}$. Such a subgroup is called a Levi subgroup. It is known that L is a closed connected subgroup. Denote $M = G/L$ and let $o \in M$ be the image of the unity by the natural projection $G \to M$. Then L is the stabilizer of o. It is known that M may be realized as a semisimple orbit of G in the coadjoint representation $\mathfrak{g}^*$. Conversely, any semisimple orbit in $\mathfrak{g}^*$ is a quotient of G by a Levi subgroup. We call *the rank* of M the dimension of the center of $\mathfrak{l}$. So, if M is a maximal orbit, i.e., $\mathfrak{l}$ is equal to the Cartan subalgebra, then the rank of M is equal to the rank of $\mathfrak{g}$.

7.1. One parameter quantization. In [**Do3**] the following statement is proven.

THEOREM 7.1. *Let G be a simple Lie group, $\mathfrak{g}$ its Lie algebra, M a semisimple orbit in $\mathfrak{g}^*$. Then*

a) The set of all φ-brackets on M form an affine nonsingular algebraic manifold, X_M, of dimension rank(M).

b) There exists an analytic universal $U_h(\mathfrak{g})$-invariant family of multiplications $\mu_{f,h}$ on $\mathcal{A}(M)$ of the form

$$\mu_{f,h}(a,b) = ab + (h/2)(f(a,b) - r_M(a,b)) + \sum_{n\geq 2} h^n \mu_{f,n}(a,b), \quad f \in X.$$

The universality means that for any $U_h(\mathfrak{g})$-invariant multiplication, m_h, there exists a formal path in X_M, $\psi(h)$, such that m_h is equivalent to $\mu_{\psi(h),h}$, and multiplications corresponding to different paths are not equivalent.

REMARK 7.2. Let us consider M as an abstract manifold. Then all nondegenerate G-invariant Poisson brackets on it appear as restrictions of the Poisson-Lie bracket by embeddings M into $\mathfrak{g}^*$ as orbits. In the quantum case, when $\mathfrak{g} = sl(n)$, a dense open set of φ-brackets in X_M can be obtained in the same way, as restrictions of the quadratic bracket (6.1). But in general there are φ-brackets on M which can not be obtained in this way. For example, the Sklyanin-Drinfeld bracket (5.5) can be carried over from G to M considered as a quotient of G, but the obtained bracket can not be realized by embedding M into $\mathfrak{g}^*$ as an orbit.

Note that also for $\mathfrak{g} \neq sl(n)$ the essential part of φ-brackets on M may be obtained with the help of embeddings M into $\mathfrak{g}^*$ as restrictions of the bracket (5.9) carried over to an open set of $\mathfrak{g}^*$ (see Remark 6.7).

7.2. Two parameter quantization. It turns out that φ-brackets compatible with the Kirillov bracket exist not on any orbits in $\mathfrak{g}^*$.

DEFINITION 7.3. Let M be an orbit in $\mathfrak{g}^*$ (not necessarily semisimple). We call M a *good* orbit, if there exists a φ-bracket on it compatible with the Kirillov bracket (we call such a bracket a *good* bracket).

The following proposition, [**DGS**], gives a classification of good semisimple orbits.

PROPOSITION 7.4. *a) For $\mathfrak{g}$ of type A_n all semisimple orbits are good.*

b) For all other $\mathfrak{g}$, a semisimple orbit M is good if and only if $M = G/L$. Here L is a Levi subgroup whose Lie algebra is generated by the Cartan subalgebra and root vectors $X_{\pm\alpha}$, where α runs all simple roots away of one or two roots which accure in the representation of the maximal root with coefficient 1.

c) The good brackets f on a good semisimple orbit, M, form a one-dimensional variety: all such brackets have the form

$$\pm f_0 + c \cdot s_M,$$

where s_M is the Kirillov bracket on M, $c \in \mathbb{C}$, and f_0 is a fixed good bracket.

So, if $\mathfrak{g} \neq sl(n)$, the good orbits are either Hermitian symmetric spaces or, for $\mathfrak{g}$ of types D_n and E_6, bisymmetric spaces.

It follows from Proposition 7.4 c) that for a good orbit the family of Poisson brackets

$$h(f_0 - r_M) + ts_M, \tag{7.1}$$

is uniquely defined up to a linear change of parameters (h, t).

PROPOSITION 7.5. *Let M be a good semisimple orbit. Then the family (7.1) can be quantized, i.e. there exists a two parameter $U_h(\mathfrak{g})$-invariant multiplication in $\mathcal{A}(M)$ of the form*

$$m_{t,h}(a,b) = ab + \frac{1}{2}(h(f_0 - r_M) + ts_M) + \sum_{k+l\geq 2} h^k t^l m_{k,l}(a,b).$$

The theorem is proven in [**Do3**].

8. Poisson and φ-brackets on quotients of G by reductive subgroups of maximal rank

Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{h}$ a fixed Cartan subalgebra. Let $\Omega \subset \mathfrak{h}^*$ be the system of roots corresponding to $\mathfrak{h}$. For each $\alpha \in \Omega$, we fix a root vector $E_\alpha \in \mathfrak{g}$ in such a way that $(E_\alpha, E_{-\alpha}) = 1$ for the Killing form $(\cdot,\cdot)$ on $\mathfrak{g}$.

Let Γ be a subsystem of roots in Ω (may be empty). Denote by $\mathfrak{g}_\Gamma$ the subalgebra of $\mathfrak{g}$ generated by elements E_α, $\alpha \in \Gamma$, and $\mathfrak{h}$. Such $\mathfrak{g}_\Gamma$ is a reductive subalgebra in $\mathfrak{g}$ of maximal rank.

Let G be a complex connected Lie group with Lie algebra $\mathfrak{g}$ and G_Γ a subgroup with Lie algebra $\mathfrak{g}_\Gamma$. Let M be a homogeneous space of G with G_Γ being the stabilizer of a point $o \in M$, so that we can identify M with the coset space G/G_Γ.

The projection $\pi : G \to M$ induces the map $\pi_* : \mathfrak{g} \to T_o$ where T_o is the tangent space to M at the point o. Since the ad-action of $\mathfrak{g}_\Gamma$ on $\mathfrak{g}$ is semisimple, there exists an $\mathrm{ad}(\mathfrak{g}_\Gamma)$-invariant subspace $\mathfrak{m}$ of $\mathfrak{g}$ complementary to $\mathfrak{g}_\Gamma$, and one can identify T_o and $\mathfrak{m}$ by means of π_*. It is easy to see that subspace $\mathfrak{m}$ is uniquely defined and has a basis formed by elements E_γ, $\gamma \in \Omega \setminus \Gamma$.

We will use the following notations. Let $\mathbb{Z}(\Omega)$ be the lattice in $\mathfrak{h}^*$ generated by Ω and $\mathbb{Z}(\Gamma)$ its sub-lattice generated by Γ. Denote by $\overline{\Omega}$ the image of Ω in $\mathbb{Z}(\Omega)/\mathbb{Z}(\Gamma)$ without zero and call elements of $\overline{\Omega}$ *quasiroots*. Since the subgroup $\mathbb{Z}(\Omega)/\mathbb{Z}(\Gamma)$ is generated by $\overline{\Omega}$, we denote $\mathbb{Z}(\overline{\Omega}) = \mathbb{Z}(\Omega)/\mathbb{Z}(\Gamma)$.

REMARK 8.1. In general, $\mathbb{Z}(\overline{\Omega})$ is not a free group. Moreover, one can prove that $\mathbb{Z}(\overline{\Omega})$ is free if and only if M can be realized as a coadjoint orbit; in this case $\mathfrak{g}_\Gamma$ is a Levi subalgebra in $\mathfrak{g}$.

Also, one can prove that any element of $\mathrm{Tor}(\mathbb{Z}(\overline{\Omega}))$ is a quasiroot.

It follows from the above that invariant bivector fields on M correspond to $\mathfrak{g}_\Gamma$-invariant bivectors of $\wedge^2\mathfrak{m}$. It is proven in [**DGS**] that any bivector of $\wedge^2\mathfrak{m}$ is $\mathfrak{h}$-invariant if and only if it has the form

$$\sum c(\bar{\alpha}) E_\alpha \wedge E_{-\alpha}, \tag{8.1}$$

where the sum runs over $\alpha \in \Omega \setminus \Gamma$, $\bar{\alpha}$ is the image of α in $\overline{\Omega}$, and coefficients $c(\bar{\alpha})$ satisfy the condition $c(\bar{\alpha}) = -c(-\bar{\alpha})$. This means that for invariant bivectors the coefficients before terms $E_\alpha \wedge E_{-\alpha}$ and $E_\beta \wedge E_{-\beta}$ are the same if $\bar{\alpha} = \bar{\beta}$.

Let M be a quotient of G by a reductive subgroup of maximal rank. The following description of φ-brackets on M, can by obtained using methods of [**DGS**] and [**Os**] (see also [**Do3**] where it is done for semisimple coadjoint orbits).

PROPOSITION 8.2. *A φ-bracket on M is determined by the following data: choosing a subgroup $\Psi \subset \mathbb{Z}(\overline{\Omega})$ containing $\mathrm{Tor}(\mathbb{Z}(\overline{\Omega}))$, a group homomorphism $\lambda : \Psi \to \mathbb{C}/2\pi i\mathbb{Z}$ such that $\lambda(\bar{\alpha}) \neq 0$ for $\bar{\alpha} \in \overline{\Omega}$, and a linear ordering of the free group $\mathbb{Z}(\overline{\Omega})/\Psi$. The coefficients $c(\bar{\alpha})$, $\bar{\alpha} \in \overline{\Omega}$, of the respective φ-bracket of the form (8.1) are defined as follows:*

$$c(\bar{\alpha}) = \frac{e^{\lambda(\bar{\alpha})} + 1}{e^{\lambda(\bar{\alpha})} - 1} \qquad \text{for } \bar{\alpha} \in \Psi,$$

$$c(\bar{\alpha}) = 1 \qquad \text{if the projection of } \bar{\alpha} \text{ in } \mathbb{Z}(\overline{\Omega})/\Psi \text{ is positive,}$$

$$c(\bar{\alpha}) = -1 \qquad \text{if the projection of } \bar{\alpha} \text{ in } \mathbb{Z}(\overline{\Omega})/\Psi \text{ is negative.}$$

Using the same methods one can obtain the following description of Poisson brackets on M.

PROPOSITION 8.3. *A Poisson bracket on M is determined by choosing a subgroup $\Psi \subset \mathbb{Z}(\overline{\Omega})$ containing no torsion elements and a group homomorphism $\lambda : \Psi \to \mathbb{C}$ such that $\lambda(\bar{\alpha}) \neq 0$ for $\bar{\alpha} \in \overline{\Omega}$. The coefficients $c(\bar{\alpha})$, $\bar{\alpha} \in \overline{\Omega}$, of the respective φ-bracket of the form (8.1) are defined as follows:*

$$c(\bar{\alpha}) = \frac{1}{\lambda(\bar{\alpha})} \qquad \text{for } \bar{\alpha} \in \Psi,$$

$$c(\bar{\alpha}) = 0 \qquad \text{if the projection of } \bar{\alpha} \text{ in } \mathbb{Z}(\overline{\Omega})/\Psi \text{ is not zero.}$$

REMARK 8.4. As mentioned in [**Lu**], the φ-brackets described in Proposition 8.2 closely relate to solutions of classical dynamical Yang-Baxter equations, [**EV**].

Another description of Poisson brackets on quotients of G by reductive subgroups of maximal rank is given in [**KS**].

REMARK 8.5. In [**Os**], the quantization of φ-brackets on M is constructed in case when $\mathbb{Z}(\overline{\Omega})$ is a finite cyclic group.

An open problem is to construct $U_h(\mathfrak{g})$-invariant quantizations for all φ-brackets on quotients of G by reductive subgroups of maximal rank.

9. Some open problems

An open question is to investigate the problem of one and two parameter quantization for non-semisimple orbits, in particular, for nilpotent ones. In this connection, I would like to mention the following problems which seem to be purely from the theory of (simple) Lie algebras:

1. Give a classification of good orbits in case $\mathfrak{g} \neq sl(n)$ (see Remark 6.7).
2. Give a classification of φ- and good brackets for non-semisimple orbits (see the same Remark).
3. Prove the following

CONJECTURE 9.1. *Let $\mathfrak{g}$ be a simple Lie algebra. Then all invariant Poisson brackets on the polynomial algebra on $\mathfrak{g}^*$, $S\mathfrak{g}$, have the form $b[\cdot,\cdot]$, where $[\cdot,\cdot]$ is the Poisson-Lie bracket and b an invariant element of $S\mathfrak{g}$.*

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Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn
Current address: Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel
E-mail address: donin@macs.biu.ac.il

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$\mathcal{U}_q(sl(n))$-covariant quantization of symmetric coadjoint orbits via reflection equation algebra

J. Donin and A. Mudrov

ABSTRACT. We study relations between the two-parameter $\mathcal{U}_q(sl(n))$-covariant deformation quantization on $sl^*(n)$ and the reflection equation algebra. The latter is described by a quantum permutation on $\mathrm{End}(\mathbb{C}^n)$ given explicitly. The reflection equation algebra is used for constructing the one-parameter quantization on coadjoint orbits, including symmetric, certain bisymmetric and nilpotent ones. Our approach is based on embedding the quantized function algebras on orbits into the algebra of functions on the quantum group $SL_q(n)$ by means of reflection equation algebra characters.

1. Introduction

This paper is devoted to a particular case of the following problem. Let M be a Poisson manifold with an action of a Lie group G. Let $\mathfrak{g}$ be the Lie algebra of the group G and $\mathcal{U}_h(\mathfrak{g})$ the corresponding quantized universal enveloping algebra, [**D**]. The problem is to construct a quantization $\mathcal{A}_h$ of the function algebra $\mathcal{A}$ on M covariant with respect to the action of $\mathcal{U}_h(\mathfrak{g})$. In this paper, we consider the case $\mathfrak{g} = sl(n)$ or $gl(n)$ and M is either $\mathfrak{g}^*$ or an orbit in $\mathfrak{g}^*$. We quantize the algebra of polynomial functions on M and present it as a quotient of the corresponding tensor algebra by very natural relations, which are similar to those in the classical case. We prove the flatness of the deformations obtained. In that way, we quantize all symmetric and some bisymmetric orbits. Note that the analogous relations can be written for all semisimple orbits but there arises the question whether the quotient algebras by those relations are flat.

Let us recall some facts about the quantization on $\mathfrak{g}^*$ and its orbits. It was shown in [**Do**] that in the case $\mathfrak{g} = sl(n)$ there exists a two-parameter deformation of the polynomial algebra $\mathrm{S}(\mathfrak{g})$ on $\mathfrak{g}^*$ which can be viewed as a $\mathcal{U}_h(\mathfrak{g})$-covariant quantization of the Lie-Poisson bracket on $\mathfrak{g}^*$. Recall that in the classical case a natural one-parameter $\mathcal{U}(\mathfrak{g})$-covariant quantization of $\mathrm{S}(\mathfrak{g})$ is given by the family $\mathrm{S}_t(\mathfrak{g}) = \mathrm{T}(\mathfrak{g})[t]/J_t$, where $\mathrm{T}(\mathfrak{g})$ is the tensor algebra of the vector space $\mathfrak{g}$ and the ideal J_t is generated by the elements $x \otimes y - \tau(x \otimes y) - t[x, y]$, $x, y \in \mathfrak{g}$. Here, τ is the flip operator on $\mathfrak{g}^{\otimes 2}$. So, the algebra $\mathrm{S}_t(\mathfrak{g})$ is quadratic-linear. By the

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Poincaré-Birkhoff-Witt theorem, $\mathrm{S}_t(\mathfrak{g})$ is a free module over $\mathbb{C}[t]$. As was proven in [**Do**], [**Do1**], that picture can be extended to the quantum case for $\mathfrak{g} = sl(n)$. Namely, there exist deformations τ_h and $[\,\cdot,\cdot\,]_h$ of both maps τ and $[\,\cdot,\cdot\,]$ such that the ideal $J_{h,t}$ generated by the elements $x \otimes y - \tau_h(x \otimes y) - t[x,y]_h$, $x, y \in \mathfrak{g}$, gives the two-parameter $\mathcal{U}_h(\mathfrak{g})$-covariant quantization $\mathrm{S}_{h,t}(\mathfrak{g}) = \mathrm{T}(\mathfrak{g})[[h]][t]/J_{h,t}$ of the Lie-Poisson bracket on $\mathfrak{g}^*$. It was also shown that such a family does not exist for $\mathfrak{g} \neq sl(n)$.

It is possible to prove that $\mathrm{S}_{h,t}(g)$ can be restricted to any semisimple orbit to provide a two-parameter $\mathcal{U}_h(\mathfrak{g})$-covariant quantization on it. On the other hand, one can obtain a one-parameter covariant quantization on a semisimple orbit M as the subalgebra of quantized functions on G invariant under the action of the quantized stabilizer, [**DoG**]. Note that none of these statements imply any explicit form of the quantized algebras. In this paper, we explicitly describe $\mathrm{S}_{h,t}(\mathfrak{g})$ and the quantum symmetric orbits in $\mathfrak{g}^*$. We also describe quantizations of certain nilpotent and bisymmetric orbits. We conjecture that all bisymmetric orbits can be quantized within our approach.

The paper is organized as follows. Sections 2 and 3 contain some basic material about the quantum group $\mathcal{U}_q(\mathfrak{g})$, $\mathfrak{g} = sl(n)$, and its fundamental representation in the vector space $V = \mathbb{C}^n$. In this paper, we work with the quantum group $\mathcal{U}_q(\mathfrak{g})$ in the sense of Lusztig, [**Lu**], instead of $\mathcal{U}_h(\mathfrak{g})$. The latter may be considered as the completion of $\mathcal{U}_q(\mathfrak{g})$ at the point $q = 1$. Correspondingly, instead of $\mathrm{S}_{h,t}(\mathfrak{g})$ we deal with the two-parameter family $\mathrm{S}_{q,t}(\mathfrak{g})$ as a free module over $\mathbb{C}(q)$. In Section 4, we study two different algebra structures on $\mathrm{T}(V) \otimes \mathrm{T}(V^*)$ covariant with respect to the actions of either $\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$ or $\mathcal{U}_q(\mathfrak{g})$. As a result, we come to inequivalent embeddings of $\mathrm{T}(V \otimes V^*)$ into $\mathrm{T}(V) \otimes \mathrm{T}(V^*)$. That, in its turn, leads to different quantizations of the polynomial algebra on $V \otimes V^*$ studied in Sections 5 and 6. We prove that they are the reflection equation (RE) algebra (see [**KSkl**]) and the algebra of functions on the quantum group known as the FRT algebra (after Faddeev, Reshetikhin, and Takhtajan), [**FRT**]. It turns out that both algebras can be described uniformly by quantum permutations on $(V \otimes V^*)^{\otimes 2}$ but within different categories. In Section 7 we evaluate the lowest weight vectors of the irreducible $\mathcal{U}_q(\mathfrak{g})$-modules in $\mathcal{M}^{\otimes 2}$, where $\mathcal{M}$ is the algebra of $n \times n$ complex matrices. Using this information, we derive the involutive permutation τ_{RE} giving the commutation relations in the reflection equation algebra. That is done in Section 8. In Section 9, we reduce this permutation to the submodule $\mathfrak{g}^{\otimes 2} \subset \mathcal{M}^{\otimes 2}$, $\mathfrak{g} = sl(n)$.

In Section 7, we formulate a relation between $\mathrm{S}_{q,t}(\mathfrak{g})$ and the RE algebra, following [**Do1**]. That relation involves the reduction of τ_{RE} from $\mathcal{M}$ to $\mathfrak{g}$, and a quantum deformation of the Lie commutator $\mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ determined by τ_{RE} up to a scalar factor. Specifying values of the quantum trace in the RE algebra, we obtain one-parameter subfamilies of $\mathrm{S}_{q,t}(\mathfrak{g})$ as quotients of the RE algebra.

Section 11 is devoted to the quantization on coadjoint orbits in $\mathfrak{g}^*$. We work, actually, with the adjoint orbits in the matrix space using the isomorphism between adjoint and coadjoint modules and employing the relation between the RE algebra and the two-parameter family $\mathrm{S}_{q,t}(\mathfrak{g})$.

The RE algebra appeared as an abstraction of algebraic constructions, [**AFS**], [**KSkl**], arising from the theory of integrable models, [**Cher**]. It is also related to the braid group of a solid handlebody, [**K**]. In this paper, we consider the RE algebra as the $\mathcal{U}_q(\mathfrak{g})$-covariant quantization of the polynomial algebra on the matrix

space. It is generated by elements arranged in the matrix L and subject to a set of quadratic relations called the reflection equation.

The conjugation transformation of L with T, the generating matrix of the FRT algebra whose entries commute with the entries of L, is again an RE matrix, [**KS**]. It follows from this observation that any character of the RE algebra specifies a homomorphism to the FRT algebra. This is done in the same way as in the classical geometry, where maximal ideals of the function algebra on M correspond to points of M. Each point defines an embedding of the function algebra on the orbit into that on the group G. We use this idea for constructing quantized manifolds as quotient spaces of the quantized group. Characters of the RE algebra are exactly solutions to the *numerical* reflection equation, [**KSS**]. We find such solutions in the form of projectors of rank $k < n$. Thus, we present the one-parameter quantizations on symmetric orbits as quotients of the RE algebra. Simultaneously, they turns out to be subalgebras in the quantized function algebra on the group, and this proves flatness of the quantizations. There are also solutions other than projectors, corresponding to different paths in the parameter space of the two-parameter quantization $\mathrm{S}_{h,t}(\mathfrak{g})$.

We prove that, for the standard quantum linear groups, there is an epimorphism from the RE algebra defined on $n \times n$ matrices onto the RE algebra defined on $k \times k$ matrices for $k < n$. That enables us to build new solutions to the matrix RE by embedding a given solution into a bigger matrix as the left upper block and extending it to the whole matrix with zeros. In particular, starting from a semisimple non-degenerate RE matrix we gain an additional, zero eigenvalue. We use this method for constructing examples of quantized bisymmetric orbits, i.e., consisting of matrices with three eigenvalues. The complete classification of solutions to the matrix RE is unknown. In particular, it is interesting to find nilpotent matrices providing quantization of nilpotent orbits. We managed to build such solutions among the matrices whose square is equal to zero.

2. The quantum universal enveloping algebra $\mathcal{U}_q(sl(n))$

By $\mathfrak{g}$ we mean the complex Lie algebra $sl(n)$, and $\mathcal{U}_q(\mathfrak{g})$ is the corresponding quantum group. The latter is understood in the sense of Lusztig [**Lu**], i.e., a free module over the field of rational functions in q. In the classical limit $q \to 1$, $\mathcal{U}_q(\mathfrak{g})$ turns into the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. The quantum group $\mathcal{U}_q(\mathfrak{g})$ is generated by the elements H_i, $X_i^{\pm}$, $i = 1, \dots, n-1$, satisfying the commutation relations

$$[H_i, X_i^{\pm}] = \pm 2X_i^{\pm}, \quad [H_i, X_{i\pm1}^{\pm}] = \mp X_{i\pm1}^{\pm},$$
$$[H_i, X_j^{\pm}] = 0, \quad |i-j| > 1,$$
$$[X_i^{+}, X_j^{-}] = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}.$$

Besides, the Serre relations hold:

$$(X_i^{\kappa})^2 X_{i\pm1}^{\kappa} - 2_q X_i^{\kappa} X_{i\pm1}^{\kappa} X_i^{\kappa} + X_{i\pm1}^{\kappa} (X_i^{\kappa})^2 = 0, \quad [X_i^{\kappa}, X_j^{\kappa}] = 0, \ |i-j| > 1,$$

where $\kappa = \pm$. Here $2_q = q + q^{-1}$. In general, the quantum integer numbers are defined as $n_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. The coproduct Δ, counit ε, and antipode γ are

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta(X_i^{\pm}) = X_i^{\pm} \otimes q^{-\frac{H_i}{2}} + q^{\frac{H_i}{2}} \otimes X_i^{\pm}.$$

$$\varepsilon(H_i) = \varepsilon(X_i^{\pm}) = 0, \quad \gamma(H_i) = -H_i, \quad \gamma(X_i^{\pm}) = -q^{\mp 1} X_i^{\pm}.$$

In the (completed) tensor square of $\mathcal{U}_q(\mathfrak{g})$, there is an element $\mathcal{R}$ called the universal R-matrix. It satisfies the conditions

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{2.1}$$

$$\mathcal{R}\Delta(x) = \Delta'(x)\mathcal{R}, \quad x \in \mathcal{U}_q(\mathfrak{g}). \tag{2.2}$$

The subscripts distinguish the legs of tensor objects and the prime stands for the opposite coproduct. We adopt the Sweedler symbolic notations $\Delta(x) = x_{(1)} \otimes x_{(2)}$ for the coproduct and denote the antipode by the bar: $\gamma(x) = \bar{x}$, $x \in \mathcal{U}_q(\mathfrak{g})$. The exact expression for $\mathcal{R}$ can be found in [**Ros, KR, KT**]. We do not use it in the sequel, rather the image of $\mathcal{R}$ in the fundamental representation on $\mathbb{C}^n$.

3. Fundamental representation of $\mathcal{U}_q(sl(n))$

By $\mathcal{M}$ we denote the algebra of $n \times n$ complex matrices with the multiplication $e_j^i e_l^k = e_l^i \delta_j^k$ defined on the matrix units e_j^i. Here δ_j^k stand for the Kronekker symbol. The elements of $\mathcal{M}$ are considered as right endomorphisms of $V = \mathbb{C}^n$ with the action $e_i e_l^k = e_l \delta_i^k$ on the basis elements $e_i \in V$. Formulas

$$\rho(H_i) = e_i^i - e_{i+1}^{i+1}, \quad \rho(X_i^+) = e_{i+1}^i, \quad \rho(X_i^-) = e_i^{i+1}, \quad i = 1, \dots, n-1.$$

define a homomorphism ρ of the algebras $\mathcal{U}_q(\mathfrak{g})$ into $\mathcal{M}$. On the Chevalley generators, it is given by the same formulas as in the classical limit $q \to 1$.

The homomorphism ρ defines the right *adjoint* and left *coadjoint* actions of $\mathcal{U}_q(\mathfrak{g})$ on $\mathcal{M}$:

$$A \triangleleft \mathrm{ad}_\rho(x) = \rho(\overline{x_{(1)}}) A \rho(x_{(2)}), \tag{3.1}$$

$$\mathrm{ad}_\rho^*(x) \triangleright A = \rho(x_{(2)}) A \rho(\overline{x_{(1)}}) \tag{3.2}$$

for $x \in \mathcal{U}_q(\mathfrak{g})$ and $A \in \mathcal{M}$. These actions are conjugate via the trace pairing on $\mathcal{M}$:

$$\mathrm{Tr}\Big(\big(A \triangleleft \mathrm{ad}_\rho(x)\big)\, B \Big) = \mathrm{Tr}\Big(A\, \big(\mathrm{ad}_\rho^*(x) \triangleright B\big) \Big).$$

The left coadjoint module $\mathcal{M}$ equipped with action (3.2) will be denoted $\mathcal{M}^*$.

The element

$$R = q^{-\frac{1}{n}} \Big(q \sum_i e_i^i \otimes e_i^i + \sum_{i \neq j} e_i^i \otimes e_j^j + \omega \sum_{i<k} e_i^k \otimes e_k^i \Big) \in \mathcal{M} \otimes \mathcal{M}, \tag{3.3}$$

where $\omega = q - q^{-1}$, satisfies the Yang-Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.$$

It differs from the R-matrix used in [**FRT**] by the overall factor $q^{-\frac{1}{n}}$. In this form, this is the image of the universal R-matrix $\mathcal{R}$ under the homomorphism $\rho^{\otimes 2}$. Let $\sigma_{V \otimes V}$ be the ordinary (classical) permutation on $V \otimes V$. The braid matrix $S = q^{\frac{1}{n}} \sigma_{V \otimes V} R$ satisfies the Hecke condition $S^2 = \omega S + 1$ and is represented by the sum of two orthogonal projectors

$$P_q^+ = \frac{1}{2_q}(q - S), \qquad P_q^- = \frac{1}{2_q}(q^{-1} + S), \tag{3.4}$$

The matrix S commutes with all elements $(\rho \otimes \rho)\Delta(x)$, $x \in \mathcal{U}_q(\mathfrak{g})$.

Let V^* be the space of linear functionals on V. We choose a basis $\{e_i\}$ in V and denote $\{f^i\}$ its dual with respect to the canonical pairing between V and V^*.

PROPOSITION 3.1. *The homomorphism $\rho\colon \mathcal{U}_q(\mathfrak{g}) \to \mathcal{M}$ induces the right and left actions of $\mathcal{U}_q(\mathfrak{g})$ on the tensor algebra $\mathrm{T}(V \oplus V^*)$. They are given on the basis elements $e_i \in V$ and $f^i \in V^*$ by*

$$e_i \triangleleft x = \sum_{\alpha=1}^{n} e_\alpha \rho(x)_i^\alpha, \qquad f^i \triangleleft x = \sum_{\alpha=1}^{n} f^\alpha \rho(\bar{x})_\alpha^i, \tag{3.5}$$

$$x \triangleright e_i = \sum_{\alpha=1}^{n} e_\alpha \rho(\bar{x})_i^\alpha, \qquad x \triangleright f^i = \sum_{\alpha=1}^{n} f^\alpha \rho(x)_\alpha^i, \tag{3.6}$$

for $x \in \mathcal{U}_q(\mathfrak{g})$. Formula $\langle e_i, f^j \rangle = \delta_i^j$ defines a $\mathcal{U}_q(\mathfrak{g})$-invariant right pairing between V and V^*. The right module $V^* \otimes V$ with the action induced by (3.5) is isomorphic to the adjoint module $\mathcal{M}$ via the correspondence $f^i \otimes e_j \to e_j^i$. The left module $V \otimes V^*$ with the action induced by (3.6) is isomorphic to the coadjoint module $\mathcal{M}^*$ via the correspondence $e_j \otimes f^i \to e_j^i$.

PROOF. Direct verification. □

One can introduce the following $\mathcal{U}_q(\mathfrak{g})$-invariant involutive permutations among the elements of V and V^*.

PROPOSITION 3.2. *The involutive permutations*

$$\tau_{V\otimes V}(e_i \otimes e_j) = \begin{cases} sign(j-i)\frac{1-q^2}{1+q^2} e_i \otimes e_j + \frac{2q}{1+q^2} e_j \otimes e_j, & i \neq j, \\ e_i \otimes e_i, \ i = j, \end{cases} \tag{3.7}$$

$$\tau_{V^*\otimes V^*}(f^i \otimes f^j) = \begin{cases} sign(i-j)\frac{1-q^2}{1+q^2} f^i \otimes f^j + \frac{2q}{1+q^2} f^j \otimes f^j, \ i \neq j, \\ f^i \otimes f^i, \quad i = j, \end{cases} \tag{3.8}$$

$$\tau_{V\otimes V^*}(e_j \otimes f^i) = \begin{cases} q^{-\frac{1}{n}}(q f^i \otimes e_i + \omega \sum_{k>i} f^k \otimes e_k), \ i = j, \\ q^{-\frac{1}{n}} f^i \otimes e_j, \quad i \neq j, \end{cases} \tag{3.9}$$

are invariant with respect to the left $\mathcal{U}_q(\mathfrak{g})$-action (3.6). The permutation

$$\tau^r_{V\otimes V}(e_i \otimes e_j) = \begin{cases} sign(i-j)\frac{1-q^2}{1+q^2} e_i \otimes e_j + \frac{2q}{1+q^2} e_j \otimes e_j, & i \neq j, \\ e_i \otimes e_i, \quad i = j, \end{cases} \tag{3.10}$$

is invariant with respect to the right $\mathcal{U}_q(\mathfrak{g})$-action (3.5).

PROOF. As a lenear operator, composition of $\mathcal{R}$ acting on $V \otimes V$ and $V^* \otimes V^*$ with the ordinary flip has two eigenspaces. They are deformations of symmetric and antisymmetric 2-tensors, and involutions (3.7), (3.8), and (3.10) are defined as multiplication by ± 1 on those subspaces. Operation (3.9) is readily obtained from the action of the universal R-matrix: $\tau_{V^*\otimes V}(f\otimes e) = \mathcal{R}_2 \triangleright e \otimes \mathcal{R}_1 \triangleright f$, $\tau_{V\otimes V^*}(e\otimes f) = \mathcal{R}_1^{-1} \triangleright f \otimes \mathcal{R}_2^{-1} \triangleright e$, $e \in V$, $f \in V^*$. To compute $\tau_{V\otimes V^*}$, it is enough to know only the image (3.3) of $\mathcal{R}$ in $\mathcal{M}^{\otimes 2}$. □

4. Algebra $\mathrm{T}(V) \otimes \mathrm{T}(V^*)$ in braided categories

In this section, we consider the tensor algebras $\mathrm{T}(V)$ and $\mathrm{T}(V^*)$ from the different points of view: as objects from the category of either $\mathcal{U}_q(\mathfrak{g})$- or $\mathcal{U}_q^{\otimes 2}(\mathfrak{g})$-representations. Correspondingly, the algebra structure on $\mathrm{T}(V) \otimes \mathrm{T}(V^*)$ may be introduced in different ways, as the tensor product of algebras in the those categories. A particular choice of the category determines embedding of $\mathrm{T}(V \otimes V^*)$,

the tensor algebra of the space $V \otimes V^*$, into $\mathrm{T}(V) \otimes \mathrm{T}(V^*)$. That leads to different quantizations of the polynomial algebra on $V \otimes V^*$.

Let us remind the construction of the tensor product of algebras in the category of a quantum group modules. Let $\mathcal{H}$ be a quantum group with an R-matrix $\mathcal{R}$. Recall that a (left) $\mathcal{H}$-module algebra $\mathcal{A}$ is an associative algebra with unit in the category of $\mathcal{H}$-representations:

$$x \triangleright (ab) \quad = \quad (x_{(1)} \triangleright a)(x_{(2)} \triangleright b), \quad x \triangleright 1 = \varepsilon(x)1, \quad x \in \mathcal{U}_q(\mathfrak{g}),\ a, b \in \mathcal{A}. \tag{4.1}$$

Given two module algebras $\mathcal{A}$ and $\mathcal{B}$, one can introduce their (braided) tensor product $\mathcal{A} \check{\otimes} \mathcal{B}$, which is again an $\mathcal{H}$-module algebra.

PROPOSITION 4.1. *Let $\mathcal{R}_1 \otimes \mathcal{R}_2$ denote the decomposition of the universal R-matrix into the two tensor factors (summation supressed). The formula*

$$(a_1 \check{\otimes} b_1)(a_2 \check{\otimes} b_2) = a_1(\mathcal{R}_2 \triangleright a_2) \check{\otimes} (\mathcal{R}_1 \triangleright b_1) b_2, \quad a_i \in \mathcal{A}, \quad b_i \in \mathcal{B} \tag{4.2}$$

defines an associative multiplication on $\mathcal{A} \check{\otimes} \mathcal{B}$ turning it into an $\mathcal{H}$-module algebra. Embeddings $\mathcal{A} \to \mathcal{A} \check{\otimes} 1$ and $\mathcal{B} \to 1 \check{\otimes} \mathcal{B}$ are homomorphisms of algebras.

PROOF. Associativity of multiplication (4.2) follows from (2.1). Compatibility with the action of $\mathcal{H}$ is a consequence of (2.2). The last statement of the proposition is immediate due to the equality $(\varepsilon \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes \varepsilon)(\mathcal{R}) = 1 \otimes 1$ following from the definition of the universal R-matrix. □

Note that multiplication (4.2) is characterized by the permutation relation

$$ba = (\mathcal{R}_2 \triangleright a)(\mathcal{R}_1 \triangleright b) \tag{4.3}$$

between the elements $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

PROPOSITION 4.2. *Let V and U be $\mathcal{H}$-modules. The subalgebra in $\mathrm{T}(V) \check{\otimes} \mathrm{T}(U)$ generated by the submodule $V \check{\otimes} U$ is isomorphic to the tensor algebra $\mathrm{T}(V \otimes U)$. It is an algebra in the category of $\mathcal{H}$-modules.*

PROOF. Embedding $V \otimes U \to \mathrm{T}(V) \check{\otimes} \mathrm{T}(U)$ is extended to a homomorphism of the free algebra $\mathrm{T}(V \otimes U)$. It is an monomorphism due to invertibility of permutation (4.3). The subalgebra generated by $V \otimes U$ is $\mathcal{H}$-invariant, so it is a module algebra. □

We will use this proposition when $U = V^*$.

Let us apply the construction above to the algebras $\mathcal{A} = \mathrm{T}(V)$, $\mathcal{B} = \mathrm{T}(V^*)$ considered as $\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$- and $\mathcal{U}_q(\mathfrak{g})_{op} \otimes \mathcal{U}_q(\mathfrak{g})$-module algebras in the following way. First let us note that a right action of a Hopf algebra is the same as a left one for its opposite. Right and left actions (3.5) and (3.6) of the algebra $\mathcal{U}_q(\mathfrak{g})$ are extended to the left actions of the Hopf algebras $\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$- and $\mathcal{U}_q(\mathfrak{g})_{op} \otimes \mathcal{U}_q(\mathfrak{g})$ on V and V^*:

$$(x \otimes y) \triangleright v = \varepsilon(y) x \triangleright v, \qquad (x \otimes y) \triangleright \phi = \varepsilon(x) y \triangleright \phi, \tag{4.4}$$

where $v \in V$ and $\phi \in V^*$. These actions are extended to the actions on $\mathrm{T}(V)$, $\mathrm{T}(V^*)$, and on the tensor product $\mathrm{T}(V) \otimes \mathrm{T}(V^*)$.

PROPOSITION 4.3. *The algebra $\mathrm{T}(V) \check{\otimes} \mathrm{T}(V^*)$ in the categories of $\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$- and $\mathcal{U}_q(\mathfrak{g})_{op} \otimes \mathcal{U}_q(\mathfrak{g})$-modules has the multiplication:*

$$(a_1 \check{\otimes} b_1)(a_2 \check{\otimes} b_2) = a_1 a_2 \check{\otimes} b_1 b_2, \quad a_i \in \mathrm{T}(V), \quad b_i \in \mathrm{T}(V^*). \tag{4.5}$$

PROOF. In both cases, the universal R-matrix is the product of two $\mathcal{R}$-s supported in the two different copies of $\mathcal{U}_q(\mathfrak{g})$. By construction (4.4), each copy acts trivially either on $\mathrm{T}(V)$ or on $\mathrm{T}(V^*)$, so the universal R-matrix turns into the identity operation in (4.2). □

Note that, being considered in the category of $\mathcal{U}_q(\mathfrak{g})$-modules, the tensor product $\mathrm{T}(V)\check{\otimes}\mathrm{T}(V^*)$ has a non-classical multiplication characterized by the permutation relation (4.3), where $\mathcal{R}$ is the R-matrix for $\mathcal{U}_q(\mathfrak{g})$.

Due to Proposition 4.2, the tensor algebra $\mathrm{T}(V\otimes V^*)$ has different realizations in the categories of either $\mathcal{U}_q(\mathfrak{g})_{op}\otimes\mathcal{U}_q(\mathfrak{g})$- or $\mathcal{U}_q(\mathfrak{g})$-representations. One can introduce additional relations in $\mathrm{T}(V\otimes V^*)$ resulting in two different quantizations of the polynomial algebra on $V\otimes V^*$; they are the FRT and RE algebras. We consider them in Sections 5 and 6 and present their defining commutation relations in terms of involutive permutations τ_{FRT} and τ_{RE}.

Taking into account Proposition 4.3, we reserve the symbol $\check{\otimes}$ only for the tensor product in the category of $\mathcal{U}_q(\mathfrak{g})$-modules where the permutation (4.3) between V and V^* is non-classical. So, by $\mathrm{T}(V)\check{\otimes}\mathrm{T}(V^*)$ we will denote the tensor product considered as $\mathcal{U}_q(\mathfrak{g})$-module.

5. Algebra $\mathcal{A}_{FRT}(\mathcal{M})$

Denote $\mathcal{A}_{FRT}(\mathcal{M})$ be the associative unital algebra over $\mathbb{C}(q)$ generated by the matrix elements T^i_j subject to relations

$$ST_1T_2 \;=\; T_1T_2S. \tag{5.1}$$

Here, S is the Hecke matrix defined in Section 3. The algebra $\mathcal{A}_{FRT}(\mathcal{M})$ was introduced in [**FRT**] as a quantized polynomial algebra on the space of matrices. It is endowed with the $\mathcal{U}_q(\mathfrak{g})_{op}\otimes\mathcal{U}_q(\mathfrak{g})$-module structure coming from that on $V\otimes V^*$. Relations (5.1) are given by a quantum permutation τ_{FRT} on the space $V\otimes V^*$, which we are going to present explicitly.

Consider $\mathrm{T}(V\otimes V^*)$ as a subalgebra of $\mathrm{T}(V)\otimes\mathrm{T}(V^*)$ in the tensor category of $\mathcal{U}_q(\mathfrak{g})_{op}\otimes\mathcal{U}_q(\mathfrak{g})$-modules, along the line of Proposition 4.2. Consider the permutation

$$\tau_{FRT} \;=\; \sigma_{V\otimes V^*}\circ\tau^r_{V\otimes V}\circ\tau_{V^*\otimes V^*}\circ\sigma^{-1}_{V\otimes V^*} \tag{5.2}$$

on $(V\otimes V^*)^{\otimes 2}$, which is the composition of elementary permutations (3.8), (3.10), and the classical flip $\sigma_{V\otimes V^*}$. The permutation τ_{FRT} is $\mathcal{U}_q(\mathfrak{g})_{op}\otimes\mathcal{U}_q(\mathfrak{g})$-invariant. Let $(V\otimes V^*)^{\otimes 2} = I^+_{FRT}\oplus I^-_{FRT}$ be the decomposition into symmetric and skew-symmetric submodules with respect to τ_{FRT}. It is easy to see that I^-_{FRT} is a deformation of the exterior square $(V\otimes V^*)^{\wedge 2}$.

THEOREM 5.1. *Let $\mathcal{J}_{FRT}$ be the ideal in $\mathrm{T}(V\otimes V^*)$ generated by I^-_{FRT}. The quotient algebra $\mathrm{T}(V\otimes V^*)/\mathcal{J}_{FRT}$ is isomorphic to $\mathcal{A}_{FRT}(\mathcal{M})$.*

PROOF. Condition (5.1) is equivalent to the pair of relations $P^\pm_q T_1T_2P^\mp_q = 0$, where $P^\pm_q$ are the projectors (3.4). Let us take a basis $\{e_i\}\subset V$ and its right dual $\{f^i\}\subset V^*$, i.e., $\langle e_i, f^j\rangle = \delta^j_i$. The elements $e_i\otimes f^j$ transform as entries of the matrix T under the action of $\mathcal{U}_q(\mathfrak{g})_{op}\otimes\mathcal{U}_q(\mathfrak{g})$, so we can identify them. The product of two matrix entries can be rewritten as $T^i_jT^m_n = (e_j\otimes f^i)(e_n\otimes f^m) = e_je_n\otimes f^if^m$, since the elements of V and V^* commute. This implies that the quadratic submodule I^-_{FRT} goes over into the submodule spanned by $\sum_{\alpha,\beta,\mu\nu}(P^\pm_q)^{\alpha\beta}_{jn}e_\alpha e_\beta\otimes f^\mu f^\nu(P^\mp_q)^{im}_{\mu\nu}$,

under the embedding $T(V \otimes V^*) \to T(V) \otimes T(V^*)$. This submodule is formed by symmetric tensors with respect to the permutation $\tau^r_{V\otimes V} \circ \tau_{V^*\otimes V^*}$. □

6. Algebra $\mathcal{A}_{RE}(\mathcal{M})$

The reflection equation algebra $\mathcal{A}_{RE}(\mathcal{M})$ is generated by the entries of the matrix L subject to the quadratic relations, [**AFS, KSkl**],

$$L_2SL_2S \;=\; SL_2SL_2. \tag{6.1}$$

Here, S is the Hecke matrix introduced in Section 3. The generators L^i_j form the coadjoint module $\mathcal{M}^*$ via the correspondence $L^i_j \to e^i_j$. Action (3.2) is extended over $\mathcal{A}_{RE}(\mathcal{M})$ turning it into a $\mathcal{U}_q(\mathfrak{g})$-module algebra;

The element $Q = \mathcal{R}_{21}\mathcal{R}$ defines a morphism of $\mathcal{U}_q(\mathfrak{g})$-modules $\mathcal{A}_{RE}(\mathcal{M}) \to \mathcal{U}_q(\mathfrak{g})$ by the correspondence $L \to Q_2 \otimes \rho(Q_1)$, [**Mj**], where $\mathcal{U}_q(\mathfrak{g})$ is considered as a left $\mathcal{U}_q(\mathfrak{g})$-module via the adjoint action. The algebra $\mathcal{A}_{RE}(\mathcal{M})$ is a flat deformation of the polynomial algebra on the matrix space $\mathcal{M}$, [**Mj**].

The algebra $\mathcal{A}_{RE}(\mathcal{M})$ can be described in the same way as $\mathcal{A}_{FRT}(\mathcal{M})$ by a quantum permutation τ_{RE} . Following Proposition 4.2, consider the tensor algebra $\mathrm{T}(V \otimes V^*)$ as the subalgebra $\mathrm{T}(V\check{\otimes}V^*) \subset \mathrm{T}(V)\check{\otimes}\mathrm{T}(V^*)$ in the category of $\mathcal{U}_q(\mathfrak{g})$-modules. Introduce the involutive operation

$$\tau_{RE} \;=\; \tau_{V\otimes V^*} \circ \tau_{V\otimes V} \circ \tau_{V^*\otimes V^*} \circ \tau^{-1}_{V\otimes V^*}, \tag{6.2}$$

which is the composition of elementary permutations (3.7–3.9). The permutation τ_{RE} is $\mathcal{U}_q(\mathfrak{g})$-invariant. Consider the decomposition $(V \otimes V^*)^{\otimes 2} = I^+_{RE} \oplus I^-_{RE}$ into the symmetric and skew-symmetric submodules with respect to τ_{RE}. It is easy to see that the submodule I^-_{RE} is a deformation of the exterior square $(V \otimes V^*)^{\wedge 2}$.

THEOREM 6.1. *Let $\mathcal{J}_{RE}$ be the ideal in $\mathrm{T}(V \otimes V^*)$ generated by I^-_{RE}. The quotient algebra $\mathrm{T}(V\check{\otimes}V^*)/\mathcal{J}_{RE}$ is isomorphic to $\mathcal{A}_{RE}(\mathcal{M})$.*

PROOF. As in the proof of Theorem 5.1, let $\{f^i\} \subset V^*$ be the right dual to the basis $\{e_i\} \subset V$. Due to Proposition 3.1, the elements $e_i\check{\otimes}f^j$ form the coadjoint module $\mathcal{M}^*$ for $U_q(\mathfrak{g})$, so we can put $L^j_i = e_i\check{\otimes}f^j$. By Proposition 4.2, the algebra $\mathrm{T}(V\check{\otimes}V^*)$ is isomorphic to the free associative algebra generated by L^i_j. It remains to show that the quadratic submodule generating the ideal $\mathcal{J}_{RE}$ is isomorphic to the submodule specified by the reflection equation. They are conjugate by the $\tau_{V\otimes V^*}$, which is the restriction to $V \otimes V^*$ of permutation (4.3). Consider the equality $e_ne_i\check{\otimes}f^kf^n \otimes e^n_k \otimes e^i_m = \sigma_{V\otimes V}L_2SL_2$ in the algebra $T(V)\check{\otimes}T(V^*)$. Since $T(V)$ is considered as a left module for $\mathcal{U}_q(\mathfrak{g})$, the permutation $\tau_{V\otimes V}$ is $\sigma_{V\otimes V}$-conjugate with the permutation $\tau^r_{V\otimes V}$. Therefore, the quadratic submodule of q-symmetric tensors in $V^{\otimes 2} \otimes V^{*\otimes 2}$ is $\sigma_{V\otimes V}$-conjugate with the $\mathcal{U}_q(\mathfrak{g})_{op} \otimes \mathcal{U}_q(\mathfrak{g})$-invariant submodule providing the algebra $\mathcal{A}_{FRT}(\mathcal{M})$: $\sigma_{V\otimes V}S\sigma_{V\otimes V}\big(\sigma_{V\otimes V}L_2SL_2\big) = \big(\sigma_{V\otimes V}L_2SL_2\big)S$. This is equivalent to the reflection equation (6.1). □

REMARK 6.2. In the classical case, the polynomial algebra on $V \otimes V^*$ can be built equivalently as a $\mathcal{U}(\mathfrak{g})$- or $\mathcal{U}^{\otimes 2}(\mathfrak{g})$-module. In the quantum case, depending on the point of view, one comes to either FRT or RE algebras. It can be shown that there is a sequence of twist transformations in the quasitensor category of $\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$-modules relating algebras $\mathcal{A}_{FRT}(\mathcal{M})$ and $\mathcal{A}_{RE}(\mathcal{M})$. This is a subject of our forthcoming publication.

7. Irreducible submodules of $\mathcal{M}^{\otimes 2}$

This technical section contains information about submodules of $\mathcal{M}^{\otimes 2}$ and its dual in terms of highest and lowest weight vectors. They will be used in computing the quantum permutation τ_{RE} on $(V \otimes V^*)^{\otimes 2} \sim \mathcal{M}^{*\otimes 2}$. It is convenient to evaluate the conjugate operator τ_{RE}^* in the dual space first. For that reason, we consider three $\mathcal{U}_q(\mathfrak{g})$-module structures on $\mathcal{M}^{\otimes 2}$:

$$(A \otimes B) \triangleleft x = \rho(\overline{x_{(2)}}) A \rho(x_{(3)}) \otimes \rho(\overline{x_{(1)}}) B \rho(x_{(4)}), \tag{7.1}$$

$$(A \otimes B) \triangleleft x = \rho(\overline{x_{(1)}}) A \rho(x_{(2)}) \otimes \rho(\overline{x_{(3)}}) B \rho(x_{(4)}), \tag{7.2}$$

$$x \triangleright (A \otimes B) = \rho(x_{(2)}) A \rho(\overline{x_{(1)}}) \otimes \rho(x_{(4)}) B \rho(\overline{x_{(3)}}). \tag{7.3}$$

Actions (7.1) and (7.2) are nothing else than $\mathrm{ad}_{\rho\otimes\rho}$ and $\mathrm{ad}_\rho \otimes \mathrm{ad}_\rho$, correspondingly. The left action (7.3) is conjugate to (7.2) with respect to the ordinary trace pairing and it is just $\mathrm{ad}_\rho^* \otimes \mathrm{ad}_\rho^*$.

A basis of the lowest weight vectors of irreducible $\mathcal{U}_q(\mathfrak{g})$-submodules with respect to the action (7.3) may be calculated directly. It is:

$$\begin{aligned}
\delta_1 &= \sum_{i=1}^{n} q^{-2i+1} e_i^i \otimes e_1^n, \\
\delta_2 &= \sum_{i=1}^{n} e_1^n \otimes q^{-2i+1} e_i^i, \\
\delta_3 &= \sum_{i=1}^{n} e_1^i \otimes e_i^n, \\
\delta_4 &= \sum_{i=1}^{n} a_i e_i^n \otimes e_1^i, \quad a_i = 1, q^{-3}, \ldots, q^{-2n+3}, q^{-2n}, \\
\delta_5 &= \sum_{i,k=1}^{n} q^{-(2i-1)} q^{-(2k-1)} e_i^i \otimes e_k^k, \\
\delta_6 &= \sum_{i,k=1}^{n} a_i e_i^k \otimes e_k^i, \quad a_i = q^{-(2n-1)}, q^{-(2n-3)}, \ldots, q^{-3}, q^{-1}, \\
\delta_7 &= q e_1^n \otimes e_2^{n-1} + q^{-1} e_2^{n-1} \otimes e_1^n - e_1^{n-1} \otimes e_2^n - e_2^n \otimes e_1^{n-1}, \\
\delta_8 &= e_1^n \otimes e_1^n, \\
\delta_9 &= e_1^n \otimes e_1^{n-1} - q^{-1} e_1^{n-1} \otimes e_1^n, \\
\delta_{10} &= e_1^n \otimes e_2^n - q^{-1} e_2^n \otimes e_1^n,
\end{aligned}$$

The coefficients a_i entering δ_4 and δ_6 decrease by q^{-2} each step within the interval $i = 1, \ldots, n$. Here, we assume $n > 2$. The case $n = 2$ is simple and left to the reader.

The vectors δ_5 and δ_6 are invariant elements under action (7.3). They are symmetric with respect to the permutation τ_{RE}^*, since that is so for the invariant elements in the classical situation. The vectors δ_7 and δ_8 belong to the symmetric part, too, while δ_9 and δ_{10} are antisymmetric with respect to τ_{RE}^*. Indeed, they generate submodules of multiplicity one in $\mathcal{M}^{\otimes 2}$ and are deformations of classical submodules belonging to symmetric and antisymmetric parts, respectively. The

situations is more complicated in what concerns vectors $\delta_1, \dots, \delta_4$ because they generate the isotypic component of type $\mathfrak{g}$ and, in the limit $q = 1$, the corresponding classical component intersects the symmetric and skew-symmetric parts of $\mathcal{M}^{\otimes 2}$ with multiplicity two. So the problem of evaluating the permutation τ_{RE} reduces to calculating its restriction to the $\mathfrak{g}$-isotypic component of $\mathcal{M}^{*\otimes 2}$.

First, we compute the dual conjugate operation to τ_{RE} on the right module $\mathcal{M}^{\otimes 2}$ with action (7.2). A basis of lowest weight vectors has the dual consisting of highest weight ones; by that reason we also consider the highest weight vectors α^i and β^i of type $\mathfrak{g}$ with respect to actions (7.1) and (7.2), respectively:

$$
\begin{aligned}
\alpha^1 &= 1 \otimes e_n^1, \\
\alpha^2 &= \sum_{i=1}^n a_i e_n^1 \otimes e_i^i, \quad a_i = q, 1, \dots, 1, q^{-1}, \\
\alpha^3 &= \sum_{i=1}^n a_i e_i^1 \otimes e_n^i, \quad a_i = q, 1, \dots, 1, q^{-1}, \\
\alpha^4 &= \sum_{i=1}^n a_i e_n^i \otimes e_i^1, \quad a_i = 1, \dots, 1, q^{-1}, \\
\beta^1 &= 1 \otimes e_n^1, \\
\beta^2 &= e_n^1 \otimes 1, \\
\beta^3 &= \sum_{i=1}^n a_i e_i^1 \otimes e_n^i, \quad a_i = q^{-(2n-1)}, q^{-(2n-3)}, \dots, q^{-3}, q^{-1}, \\
\beta^4 &= \sum_{i=1}^n a_i e_n^i \otimes e_i^1, \quad a_i = q, 1, \dots, 1, q^{-1}.
\end{aligned} \tag{7.4}
$$

Let us denote the isotypic $\mathfrak{g}$-type components with respect to actions (7.1), (7.2), and (7.3) by $\mathcal{M}_\alpha^{\otimes 2}$, $\mathcal{M}_\beta^{\otimes 2}$, and $\mathcal{M}_\delta^{*\otimes 2}$. They are generated by the vectors α^i, β^i, and δ_i, $i = 1, \dots, 4$, respectively.

8. The quantum permutation τ_{RE}

In this section, we calculate the permutation τ_{RE} in terms of the lowest weight vectors introduced in the previous section. The permutations τ_{FRT} and τ_{RE} define quadratic relations in the deformed algebras of functions on matrices. In both cases, relations (5.1) and (6.1) are formulated using the algebra structure on $\mathcal{M}^{\otimes 2}$, i.e., in the dual setting. Thus, the FRT and RE algebra permutations are introduced through the dual operations τ_{FRT}^* and τ_{RE}^* conjugate to τ_{FRT} and τ_{RE} by the trace pairing. The involutions τ_{FRT}^* and τ_{RE}^* are invariant with respect to right actions (7.1) and (7.2). It is natural to compute them first, using the algebra structure on $\mathcal{M}^{\otimes 2}$, and then evaluate τ_{RE} by duality.

The subspace of $\Omega \in \mathcal{M}^{\otimes 2}$ such that $S\Omega = \Omega S$ is the annulator of the submodule defining FRT relations (5.1). Therefore, the involution τ_{FRT}^* is determined by the conjugation transformation with the Hecke matrix S. Solutions to the equation $S\sigma(\Omega) = \sigma(\Omega)S$, where the map σ is defined as $\Omega \to (1 \otimes \Omega_1)S(1 \otimes \Omega_2)$, $\Omega = \Omega_1 \otimes \Omega_2 \in \mathcal{M}^{\otimes 2}$, form the submodule annihilating the RE relations. Hence, the involutions τ_{RE}^* and τ_{FRT}^* are σ-conjugate: $\tau_{RE}^* = \sigma^{-1}\tau_{FRT}^*\sigma$. Since the Hecke

matrix S is invariant under the adjoint action $\mathrm{ad}_{\rho\otimes\rho}$, the map σ intertwines actions (7.2) and (7.1). It is an isomorphism of modules, being a deformation of the classical flip $\sigma_{\mathcal{M}\otimes\mathcal{M}}$.

As was noted in Section 7, only the restriction of τ^*_{RE} to the isotypic $\mathfrak{g}$-type component requires a special consideration. Restricted to the $\mathfrak{g}$-component $\mathcal{M}^{\otimes 2}_{\alpha}$, the conjugation operation with the matrix S has eigenvalues $+1$ and $-q^{\pm 2}$. The eigenspace $\mathcal{M}^{\otimes 2}_{\alpha;+}$ corresponding to the eigenvalue $+1$ consists of symmetric tensors, in the sense of the permutation τ^*_{FRT}. Eigenvectors of the eigenvalues $-q^{\pm 2}$ may be called τ^*_{FRT}-antisymmetric. We denote this submodule $\mathcal{M}^{\otimes 2}_{\alpha;-}$. Thus, the conjugation with the matrix S determines decomposition

$$\mathcal{M}^{\otimes 2}_{\alpha} = \mathcal{M}^{\otimes 2}_{\alpha;+} \oplus \mathcal{M}^{\otimes 2}_{\alpha;-}, \tag{8.1}$$

where both $\mathcal{M}^{\otimes 2}_{\alpha;\pm}$ contain irreducible submodule $\mathfrak{g}$ with multiplicity two. Decomposition (8.1) induces decompositions

$$\mathcal{M}^{\otimes 2}_{\beta} = \mathcal{M}^{\otimes 2}_{\beta;+} \oplus \mathcal{M}^{\otimes 2}_{\beta;-}, \tag{8.2}$$

$$\mathcal{M}^{*\otimes 2}_{\delta} = \mathcal{M}^{*\otimes 2}_{\delta;+} \oplus \mathcal{M}^{*\otimes 2}_{\delta;-}. \tag{8.3}$$

into the symmetric and antisymmetric parts. The subspaces $\mathcal{M}^{\otimes 2}_{\beta;\pm}$ are the images of $\mathcal{M}^{\otimes 2}_{\alpha;\pm}$ via the inverse transformation σ^{-1}. They yield the involution τ^*_{RE}. The subspaces $\mathcal{M}^{*\otimes 2}_{\delta;\pm}$ are the annulators of $\mathcal{M}^{\otimes 2}_{\beta;\mp}$ with respect to the trace pairing between $\mathcal{M}^{\otimes 2}$ and $\mathcal{M}^{*\otimes 2}$. The permutation τ_{RE} is determined by the submodules $\mathcal{M}^{*\otimes 2}_{\delta;\pm}$.

PROPOSITION 8.1. *The $\mathfrak{g}$-type submodules of τ_{RE}-symmetric and antisymmetric tensors are generated by the following highest and lowest weight vectors:*

$$\mathcal{M}^{\otimes 2}_{\alpha;+} : \alpha^3 + \alpha^4, \quad \alpha^1 + \alpha^2 + \omega\alpha^3, \qquad \mathcal{M}^{\otimes 2}_{\alpha;-} : q^{\pm 1}\alpha^1 - q^{\mp 1}\alpha^2 \mp \alpha^3 \pm \alpha^4,$$

$$\mathcal{M}^{\otimes 2}_{\beta;+} : \beta^1 + \beta^2 - \omega\beta^3, \quad \beta^3 + \beta^4 + \omega\beta^1, \quad \mathcal{M}^{\otimes 2}_{\beta;-} : \beta^1 - \beta^2, \quad \beta^3 - \beta^4,$$

$$\mathcal{M}^{*\otimes 2}_{\delta;+} : \delta_3 + \delta_4 - \omega\delta_2, \quad \delta_1 + \delta_2 + \omega\delta_3, \qquad \mathcal{M}^{*\otimes 2}_{\delta;-} : \delta_1 - \delta_2, \quad \delta_3 - \delta_4.$$

PROOF. The eigenvectors of the conjugation by S restricted to $\mathcal{M}^{\otimes 2}_{\alpha}$ are computed directly. Thus we obtain (8.1). The transformation σ acts on the highest weight vectors of the $\mathfrak{g}$-type colmponents as

$$\sigma : \begin{pmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \\ \beta^4 \end{pmatrix} \to \begin{pmatrix} \alpha^3 \\ \alpha^4 + \omega\alpha^1 \\ \alpha^1 \\ \alpha^2 \end{pmatrix}.$$

Decomposition (8.2) is the σ-preimage of (8.1). Decomposition (8.3) is computed by pairing δ_i with β^k via the ordinary trace. □

COROLLARY 8.2. *The submodule of τ_{RE}-symmetric tensors in $\mathcal{M}^{*\otimes 2}$ is generated by the lowest weight vectors*

$$\{\delta_1 + \delta_2 + \omega\delta_3, \delta_3 + \delta_4 - \omega\delta_2, \delta_5, \delta_6, \delta_7, \delta_8\}$$

The submodule of τ_{RE}-antisymmetric tensors in $\mathcal{M}^{*\otimes 2}$ is generated by the lowest weight vectors

$$\{\delta_1 - \delta_2, \delta_3 - \delta_4, \delta_9, \delta_{10}\}$$

PROOF. Immediate from the last proposition. □

The permutation τ_{RE} can be reduced to the submodule $\mathfrak{g}^{\otimes 2} \subset \mathcal{M}^{*\otimes 2}$. We study this problem in the next section.

9. Reducing τ_{RE} to $\mathfrak{g}^{\otimes 2}$

The left coadjoint module $\mathcal{M}^*$ contains a one-dimensional submodule $\mathfrak{m}_0$, which is spanned by the invariant element $\sum_{i=1}^n q^{-2i+2} e_i^i$ and the $\mathfrak{g}$-type submodule of q-traceless matrices (definition (10.2) of the quantum trace will be given in the next section). The decomposition $\mathcal{M}^* = \mathfrak{m}_0 \oplus \mathfrak{g}$ leads to the decomposition

$$\mathcal{M}^* \otimes \mathcal{M}^* = \mathfrak{m}_0 \otimes \mathfrak{m}_0 \ \oplus \ \mathfrak{m}_0 \otimes \mathfrak{g} \ \oplus \ \mathfrak{g} \otimes \mathfrak{m}_0 \ \oplus \ \mathfrak{g} \otimes \mathfrak{g}.$$

The lowest weight vectors $\tilde{\delta}_3$ and $\tilde{\delta}_4$ belonging to $\mathfrak{g} \otimes \mathfrak{g}$ are obtained by subtracting proper linear combinations of δ_1 and δ_2 from δ_3 and δ_4:

$$\tilde{\delta}_3 = \delta_3 - \tfrac{\omega}{1-q^{-2n}}\delta_1 - \tfrac{\omega}{1-q^{-2n}}\delta_2, \quad \tilde{\delta}_4 = \delta_4 - \tfrac{\omega}{q^{2n}-1}\delta_1 - \tfrac{\omega}{1-q^{-2n}}\delta_2. \tag{9.1}$$

Now we can evaluate the reduction $\tilde{\tau}_{RE}$ of the permutation τ_{RE} to the submodule $\mathfrak{g}^{\otimes 2}$.

PROPOSITION 9.1. *The composition*

$$\mathfrak{g} \otimes \mathfrak{g} \to \mathcal{M}^* \otimes \mathcal{M}^* \xrightarrow{\tau_{RE}} \mathcal{M}^* \otimes \mathcal{M}^* \to \mathfrak{g} \otimes \mathfrak{g}, \tag{9.2}$$

where the left arrow is embedding and the right one is the projection along $\mathfrak{m}_0 \otimes \mathcal{M}^* + \mathcal{M}^* \otimes \mathfrak{m}_0$, *defines an involutive* $\mathcal{U}_q(\mathfrak{g})$*-invariant permutation* $\tilde{\tau}_{RE}$ *on* $\mathfrak{g}^{\otimes 2}$. *The lowest weight vectors*

$$\tilde{\delta}_+ = (1 - \frac{\omega q^{-n}}{n_q})\tilde{\delta}_3 + (1 + \frac{\omega q^n}{n_q})\tilde{\delta}_4, \qquad \tilde{\delta}_- = \tilde{\delta}_4 - \tilde{\delta}_3, \tag{9.3}$$

where $\tilde{\delta}_i$ *are introduced by (9.1), generate the* $\mathfrak{g}$*-type submodules in* $\mathfrak{g}^{\otimes 2}$ *of symmetric and antisymmetric tensors with respect to* $\tilde{\tau}_{RE}$, *correspondingly.*

PROOF. The first statement is immediate. It easy to check that the vector $\tilde{\delta}_+ \in \mathfrak{g}^{\otimes 2} \subset \mathcal{M}^{\otimes 2}$ is τ_{RE}-symmetric. As to the vector $\tilde{\delta}_-$, it is the image of τ_{RE}-antisymmetric vector $\delta_4 - \delta_3$ under the projection $\mathcal{M}^{*\otimes 2} \to \mathfrak{g}^{\otimes 2}$. □

COROLLARY 9.2. *The submodules of symmetric and antisymmetric tensors are generated by the sets*

$$\{\delta_6 - \frac{\omega}{1-q^{-2n}}\delta_5,\ \delta_7,\ \tilde{\delta}_+\}, \quad \textit{and} \quad \{\delta_9,\ \delta_{10},\ \tilde{\delta}_-\},$$

of lowest weight vectors, respectively.

PROOF. The vector δ_5 is the square of the invariant element $\sum_{i=1}^n q^{-2i+2} e_i^i$ spanning $\mathfrak{m}_0$. The combination $\delta_6 - \frac{\omega}{1-q^{-2n}}\delta_5$ is the Casimir element of $\mathfrak{g}^{\otimes 2} \subset \mathcal{M}^{*\otimes 2}$. The $\mathfrak{g}$-type vectors $\tilde{\delta}_\pm$ were considered in the above proposition. Concerning the other lowest weight vectors, they already lye in $\mathfrak{g}^{\otimes 2} \subset M^{*\otimes 2}$, so the proof is an immediate consequence of Corollary 8.2. □

REMARK 9.3. Unlike in the classical case, the submodule $\mathfrak{g}^{\otimes 2}$ in $M^{*\otimes 2}$ is not preserved by the permutation τ_{RE}. By this reason, $\tilde{\tau}_{RE}$ is not a simple restriction of τ_{RE} to $\mathfrak{g}^{\otimes 2}$.

As a left $\mathcal{U}_q(\mathfrak{g})$-module, the tensor square $\mathcal{M}^{*\otimes 2}$ is endowed with another permutation coming from the universal R-matrix representation: $A \otimes B \to \mathcal{R}_2 \triangleright B \otimes \mathcal{R}_1 \triangleright A, \quad A \otimes B \in \mathcal{M}^{*\otimes 2}$. It can be shown that its restriction to the $\mathfrak{g}$-component $\mathcal{M}^{*\otimes 2}_{\delta} \subset \mathcal{M}^{*\otimes 2}$ acts according to the rule

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \tilde{\delta}_3 \\ \tilde{\delta}_4 \end{pmatrix} \to \begin{pmatrix} \delta_2 \\ \delta_1 \\ \tilde{\delta}_4 \\ q^{-2n}\tilde{\delta}_3 \end{pmatrix}.$$

It is easy to see, that this operator has eigensubspaces distinct from those of τ_{RE}. With this remark, we complete the study of the RE algebra. In the remainder of the paper, we investigate relations between $\mathcal{A}_{RE}(\mathcal{M})$ and the quantization on $\mathfrak{g}^*$ and its orbits.

10. Algebra $\mathcal{A}_{RE}$ and the quantization on $\mathfrak{g}^*$

It was shown in [**Do1**] that a two-parameter $\mathcal{U}_q(\mathfrak{g})$-covariant quantization $\mathrm{S}_{q,t}(\mathfrak{g})$ of the Lie-Poisson bracket on $\mathfrak{g}^*$ exists only in the $sl(n)$ case. It is realized as the quotient algebra of $\mathrm{T}(\mathfrak{g})[t](q)$ by the quadratic-linear relations

$$x \otimes y - \tilde{\tau}_{RE}(x \otimes y) \quad = \quad t[x, y]_q, \tag{10.1}$$

where $[\,\cdot,\cdot\,]_q \colon \mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ is a deformed Lie bracket. This is a $\mathcal{U}_q(\mathfrak{g})$-equivariant map $\mathfrak{g}^{\otimes 2} \to \mathfrak{g}$ sending the submodule of $\tilde{\tau}_{RE}$-symmetric tensors to zero. The commutator $[\,\cdot,\cdot\,]_q$ is uniquely defined up to a factor, since the module of $\tilde{\tau}_{RE}$-antisymmetric tensors contains the submodule isomorphic to $\mathfrak{g}$ with multiplicity one. In the limit $q \to 1$, the commutator turns into the classical Lie bracket. In this way, one recovers the algebra $U(\mathfrak{g})[t]$ viewed as a quantization of the Lie-Poisson bracket. Another limit $t \to 0$ yields a one-parameter family $\mathrm{S}_{q,0}(\mathfrak{g})$, which can be interpreted as a $\tilde{\tau}_{RE}$-commutative algebra of polynomials on $\mathfrak{g}^*$.

The RE algebra is closely related to $\mathrm{S}_{q,t}(\mathfrak{g})$. To formulate that relation, let us introduce the quantum trace of the RE matrix L:

$$\mathrm{Tr}_q(L) \quad = \quad \sum_{i=1}^{n} q^{-2i+2} L_i^i = \mathrm{Tr}(DL), \quad \text{where} \quad D = \sum_{i=1}^{n} q^{-2i+2} e_i^i. \tag{10.2}$$

Remark that this definition makes sense for matrices with entries being elements of any associative algebra $\mathcal{A}$ over $\mathbb{C}(q)$. The quantum trace is the map $\mathrm{id} \otimes \mathrm{Tr}_q \colon \mathcal{A} \otimes \mathcal{M} \to \mathcal{A} \otimes \mathbb{C}(q) \sim \mathcal{A}$. For the generating matrix L of the RE algebra, the elements $\mathrm{Tr}_q(L^k)$, $k \in \mathbb{N}$, are invariant. It follows from the commutation relations (6.1) that they belong to the center of $\mathcal{A}_{RE}(\mathcal{M})$.

THEOREM 10.1. *The quotient algebra of $\mathcal{A}_{RE}(\mathcal{M})$ by the ideal $(\mathrm{Tr}_q(L) - \lambda)$, where $\lambda \in \mathbb{C}$, is isomorphic to the one-parameter sub-family $t = \lambda \frac{q-q^{-1}}{\mathrm{Tr}_q(Id)}$ in $S_{q,t}(\mathfrak{g})$.*

PROOF. In its essential part, this theorem was proven in [**Do1**]. Here we specify the exact relation between the parameter t and the value of λ, using the explicit expression for the permutation τ_{RE} derived in Section 8, in terms of the lowest weight vectors. By Proposition 8.1, the τ_{RE}-symmetric $\mathfrak{g}$-type submodules are generated by the vectors $\delta_1 - \delta_2$ and $\delta_4 - \delta_3$. The first one is irrelevant because it turns zero when projected to $\mathfrak{g}^{\otimes 2} \subset \mathcal{M}^{*\otimes 2}$. The second vector is represented as $\delta_4 - \delta_3 = \tilde{\delta}_4 - \tilde{\delta}_3 - \omega\delta_1$, where $\tilde{\delta}_i$ belong to $\mathfrak{g}^{\otimes 2}$. Vector δ_1 is expressed through

the matrix D introduced in (10.2) and the lowest weight vector e_1^n of the adjoint representation in $\mathfrak{g}$: $\delta_1 = q^{-1}D \otimes e_1^n$. Comparing this with (10.1), we find the value of the parameter t. □

11. The quantization of coadjoint orbits

It can be shown that the two-parameter family $\mathrm{S}_{q,t}(\mathfrak{g})$ can be restricted to any semisimple orbit in $\mathfrak{g}^*$. It this connection, there arises the problem of explicit description of the quantized manifolds in terms of ideals in $\mathrm{S}_{q,t}(\mathfrak{g})$. We solve this problem (in a one-parameter setting) for certain classes of orbits including the symmetric ones, using the relation between $\mathcal{A}_{RE}$ and $\mathrm{S}_{q,t}(\mathfrak{g})$ (Theorem 10.1). Let $\mathcal{O}_A \subset \mathcal{M}$ be the orbit passing through the matrix A. Consider the two commutative diagrams

$$\begin{array}{ccc} G \times \mathcal{M} & \longrightarrow & \mathcal{M} \\ \uparrow & & \uparrow \\ G \times \{A\} & \longrightarrow & \mathcal{O}_A \end{array} \qquad \begin{array}{ccc} \mathrm{Fun}(G) \otimes \mathrm{Fun}(\mathcal{M}) & \longleftarrow & \mathrm{Fun}(\mathcal{M}) \\ \mathrm{id} \otimes \chi^A \downarrow & & \downarrow \\ \mathrm{Fun}(G) \otimes \mathbb{C} & \longleftarrow & \mathrm{Fun}(\mathcal{O}_A), \end{array}$$

where the horizontal arrows correspond to the action of the group G. The right square represents morphisms of the polynomial algebras induced by the maps of manifolds depicted on the left. The map χ^A is the character of the algebra $\mathrm{Fun}(\mathcal{M})$ corresponding to the point $A \in \mathcal{M}$. We are going to quantize the right square; that will give us realization of the quantized algebra $\mathrm{Fun}_q(\mathcal{O}_A)$, on the one hand, as a quotient of $\mathcal{A}_{RE}(\mathcal{M})$ and as a subalgebra in the quantized function algebra on G, on the other.

Let $\mathcal{F}_q(G)$ be the Hopf algebra of quantized polynomial functions on the group G. It is a quotient of the FRT algebra by the additional relation $\det_q(T) = 1$, where $\det_q$ is the quantum determinant, [**FRT**]. Since $\mathcal{U}_q(\mathfrak{g})$ acts on the RE algebra $\mathcal{A}_{RE}(\mathcal{M})$, this action generates the coaction of $\mathcal{F}_q(G)$ on $\mathcal{A}_{RE}(\mathcal{M})$.

THEOREM 11.1 ([**KS**]). *Let T^{-1} be the matrix with entries $\gamma(T_j^i)$, where T is the generating matrix of the algebra $\mathcal{F}_q(G)$ and γ the antipode. The conjugation transformation $L \to T^{-1}LT$ of an RE matrix L with the FRT matrix T whose entries commute with the entries of L is again an RE matrix.*

It follows that the correspondence $L \to T^{-1}LT$ extends to an homomorphism $\mathcal{A}_{RE}(\mathcal{M}) \to \mathcal{F}_q \otimes \mathcal{A}_{RE}(\mathcal{M})$. Theorem 11.1 yields the quantization of the upper arrow of the right square on the above diagram. To quantize the other maps, we replace χ^A by χ_q^A, a character of the reflection equation algebra. It is determined by the correspondence $L_j^i \to A_j^i \in \mathbb{C}(q)$ such that the numeric matrix A satisfies the reflection equation, [**KSS**],

$$A_2SA_2S \;=\; SA_2SA_2 \tag{11.1}$$

supported in $\mathcal{M}^{\otimes 2}$. Any solution to this equation gives rise to the algebra $\mathrm{Fun}_q(\mathcal{O}_A)$ closing the commutative diagram

$$\begin{array}{ccc} \mathcal{F}_q(G) \otimes \mathcal{A}_{RE}(\mathcal{M}) & \longleftarrow & \mathcal{A}_{RE}(\mathcal{M}) \\ \mathrm{id} \otimes \chi_q^A \downarrow & & \downarrow \\ \mathcal{F}_q(G) \otimes \mathbb{C}(q) & \longleftarrow & \mathrm{Fun}_q(\mathcal{O}_A) \end{array}$$

so that the bottom arrow is embedding. Thus, we obtain

THEOREM 11.2. *Let A be a solution of the numeric RE (11.1). Then the algebra $Fun_q(\mathcal{O}_A)$ in the diagram above is the quantization of the polynomial algebra on the orbit $\mathcal{O}_A$ passing through the matrix A.*

PROOF. It is clear that at $q = 1$ the algebra $\mathrm{Fun}_q(\mathcal{O}_A)$ coincides with the polynomial algebra on the orbit. The flatness of $\mathrm{Fun}_q(\mathcal{O}_A)$ over q follows from the fact that it is simultaneously a quotient and a subalebra of the flat $\mathbb{C}(q)$-algebras $\mathcal{A}_{RE}(\mathcal{M})$ and $\mathcal{F}_q(G)$. □

The matrix $L(A) = T^{-1}AT$ possesses the following properties:

LEMMA 11.3. *For any polynomial function $\mathcal{P}$ in one variable,*

$$\mathcal{P}\big(L(A)\big) = L\big(\mathcal{P}(A)\big). \tag{11.2}$$

The quantum trace is invariant under the conjugation:

$$\mathrm{Tr}_q(L(A)) = \mathrm{Tr}_q(A). \tag{11.3}$$

PROOF. The first statement is evident. The second one is checked using the commutation relations in the algebra $\mathcal{F}_q(G)$. □

Note that solutions A of (11.1) yield quantizations which are quotients of one-parameter subfamilies in $\mathrm{S}_{q,t}(\mathfrak{g})$. Those subfamilies are defined by paths in the parameter space (q,t) crossing the axis $q = 1$ at the point $t = 0$. In particular, the solutions with $\mathrm{Tr}_q(A) = 0$ correspond to the path $t = 0$. The limit $q \to 1$ for t separated from 0 cannot be reached within our approach. In the last subsections, we consider quantizations of various types of orbits, along the line of Theorem 11.2.

11.1. Quantized symmetric orbits. To begin with, let us prove a statement relating reflection equation algebras in different dimensions. Let $\mathcal{M}(n)$ and $\mathcal{M}(k)$ be the matrix algebra of $n \times n$ and $k \times k$ matrices, $0 < k < n$. The homomorphism $\mathcal{M}(k) \to \mathcal{M}(n)$ of embedding as the left upper block induces a contravariant epimorphism of the function algebras. The same holds in the quantum situation:

PROPOSITION 11.4. *The quotient algebra of $\mathcal{A}_{RE}\big(\mathcal{M}(n)\big)$ by the relations $L^i_j = 0$, $i > k$ or $j > k$, is isomorphic to $\mathcal{A}_{RE}\big(\mathcal{M}(k)\big)$.*

PROOF. Let P^+ and P^- be the projectors from $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{n-k}$ to the first and the second addends, respectively. Denote R^{++} the projection of the R-matrix (3.3) to $\mathcal{M}(k) \otimes \mathcal{M}(k)$. Up to a nonzero scalar factor, this is the R-matrix in dimension k. The equalities

$$(P^+ \otimes 1)R = R^{++} + q^{-\frac{1}{n}} P^+ \otimes P^-, \tag{11.4}$$

$$R(1 \otimes P^+) = R^{++} + q^{-\frac{1}{n}} P^- \otimes P^+ \tag{11.5}$$

follows directly from (3.3). The matrix L is equal to $L^+ = P^+ L P^+$ modulo the ideal specified in the hypothesis. In terms of the matrix R instead of the Hecke matrix S, the reflection equation is rewritten as

$$R_{21} L_1^+ R L_2^+ = L_2^+ R_{21} L_1^+ R$$

modulo the relations of concern. Using (11.4) and (11.5), the matrix R can be replaced by R^{++} thus leading to the RE in dimension k. □

Proposition 11.4 suggests a method of building solutions to (11.1) by those in smaller dimensions extending them by zero matrix elements. For example, the projectors $P_k = \sum_{i=1}^{k} e_i^i$ of rank $k = 1, \dots, n$ satisfy (11.1). Indeed, each can be obtained from the $k \times k$ unit matrix (which is apparently a solution to the RE) by extending it with zeros to the $n \times n$ matrix.

Let us introduce the quantum integer numbers

$$\hat{k}_q = \sum_{i=1}^{k} q^{-2i+2} = \mathrm{Tr}_q(P_k). \tag{11.6}$$

THEOREM 11.5. *The quotient algebra of $\mathcal{A}_{RE}(\mathcal{M})$ by the relations*

$$L^2 = L, \quad Tr_q(L) = \hat{k}_q \tag{11.7}$$

yields a $\mathcal{U}_q(\mathfrak{g})$-covariant quantization $Fun_q(\mathcal{O}_{P_k})$ of the symmetric orbit $\mathcal{O}_{P_k}$ passing through the projector P_k

PROOF. Relations (11.7) follow from identities (11.2) and (11.3) as applied to $A = P_k$. The projectors P_k are stable under the right adjoint action (3.1) of $\mathcal{U}_q(sl(k)) \otimes \mathcal{U}_q(sl(n-k)) \otimes \mathcal{U}_q(sl(1))$ as a quantum subgroup in $\mathcal{U}_q(sl(n))$. Therefore, the subalgebra in $\mathcal{F}_q(G)$ generated by the RE matrix $L(P_k)$ is invariant under the right action of $\mathcal{U}_q(sl(k)) \otimes \mathcal{U}_q(sl(n-k)) \otimes \mathcal{U}_q(sl(1))$. □

REMARK 11.6. The quantum $\mathcal{O}_{P_1}$ may be obtained directly from the description of $\mathcal{A}_{RE}$ given in Section 6. Indeed, imposing the additional conditions $V^{\wedge 2} = 0$ and $V^{*\wedge 2} = 0$ one comes to the subalgebra in $\mathrm{Sym}_q(V) \check{\otimes} \mathrm{Sym}_q(V^*)$ generated by $e_i \otimes f^j$ with e_i and f^j commutative in the sense of permutations (3.7–3.9). The matrix elements $L_i^j = e_i f^j$ satisfy the equality $L_i^l L_l^j = e_i(f^l e_l) f^j = Tr_q(L) L_i^j$ following from the commutation relations. So, conditions (11.7) for the case $k = 1$ hold simultaneously in this algebra. Another description of the quantum projective space is found in [**DGK**].

As in the classical situation, we can consider solution λP_k with arbitrary $\lambda \neq 0$. This will lead to the relation $L^2 = \lambda L$ and the corresponding rescaling of the quantum trace. The resulting conditions give the deformation quantization of the orbit passing through λP_k. It may be regarded as the quantization of the same manifold but along the path $t = \lambda \omega^{\frac{\hat{k}_q}{\hat{n}_q}}$ in the two parameter space of the universal family $\mathrm{S}_{q,t}(\mathfrak{g})$ (Theorem 10.1).

11.2. The Cayley-Hamilton identity. The goal of this subsection is to exhibit correspondence between Theorem 11.5 with identities in $\mathcal{A}_{RE}(\mathcal{M})$ of the Cayley-Hamiltonian type, [**PS**]. In the algebra $\mathcal{A}_{RE}(\mathcal{M})$, the k-th powers L^k of the generating matrix form the left coadjoint module $\mathcal{M}^*$. The quantum traces $\mathrm{Tr}_q(L^k)$, $k = 1, 2, \dots$ are $\mathcal{U}_q(sl(n))$-invariant and central, (see, e.g., [**PS**]).

THEOREM 11.7 ([**PS**]). *The generating matrix of the reflection equation algebra $\mathcal{A}_{RE}(\mathcal{M})$ obeys the relation*

$$\sum_{j=0}^{n} \sigma_q^j(L)(-L)^{n-j} = 0,$$

where $\sigma_q^j(L)$ are central elements expressed through the quantum traces of powers in L by the recursive formula

$$\hat{k}_q\sigma_q^k(L) = \sum_{j=0}^{k-1}\sigma_q^j(L)(-1)^{k-j+1}\mathrm{Tr}_q(L^{k-j}), \quad \sigma_q^0(L)=1. \tag{11.8}$$

In particular, $\sigma_q^{n+1}(L)=0$.

It follows that imposing the projector condition on the matrix L specifies the quantum trace modulo the finite set of values.

COROLLARY 11.8. *The condition $L^2 = L$ on the generating matrix of the $\mathcal{A}_{RE}(\mathcal{M})$ implies the discrete set $\{\hat{0}_q, \hat{1}_q, \dots, \hat{n}_q\}$ of values for the quantum trace.*

PROOF. The statement follows from the formula (11.8) since, under the hypothesis made, one has

$$\sigma_q^{n+1}(L) = \frac{\mathrm{Tr}_q(L)\big(\mathrm{Tr}_q(L)-\hat{1}_q\big)\dots\big(\mathrm{Tr}_q(L)-\hat{n}_q\big)}{\widehat{(n+1)_q!}} = 0.$$

□

All the possible values of $\mathrm{Tr}_q(L)$ are realized, by Theorem 11.5, giving flat deformations of symmetric spaces. Note that the Cayley-Hamiltonian identity in the RE algebra was directly used for construction of the quantum sphere $\mathbb{S}_q^2$ in [**GS**].

11.3. On the quantization of non-semisimple orbits. As was already mentioned, the two-parameter quantization $\mathrm{S}_{q,t}(sl(n))$ may be "restricted" to every semisimple orbits. There are no definite assertions of that kind concerning orbits passing through nilpotent elements. In this section, we prove

THEOREM 11.9. *There exists a one-parameter $\mathcal{U}_q(\mathfrak{g})$-covariant quantization of nilpotent orbits in $\mathcal{M}$ satisfying the matrix equation $A^2 = 0$. It is a restriction of the one-parameter subfamily $t = 0$ in the universal two-parameter quantization $S_{q,t}(\mathfrak{g})$.*

PROOF. Like in the case of symmetric orbits, we seek for a solution to the numeric reflection equation, which will realize the quantized algebra as a subalgebra in $\mathcal{F}_q(G)$. Specializing to the skew-diagonal matrices,

$$A = \sum_{i=1}^{n}\lambda_i e_i^{i'}, \quad i' = n+1-i,$$

we find it in the form $\lambda_i\lambda_{i'} = \lambda^2, \quad i = 1, \dots, n$. In case of non-zero λ we come to nondegenerate matrices with two eigenvalues $\pm\lambda$. These solutions were found in [**KSS**]. They lead to other realization of quantum symmetric orbits than by means of projectors. In the case of $\lambda = 0$, these matrices being squared are zero. By re-enumerating the basis elements, they can be brought to the sum of 2×2 jordanian blocks yielding all such matrices. The last statement of the theorem holds because $\mathrm{Tr}_q(A) = 0$. □

11.4. On the quantization of bisymmetric orbits. To quantize the symmetric orbits in g^*, we used the projectors, i.e., the semisimple elements with the eigenvalues 1 and 0. There are non-degenerate semisimple solutions to the matrix reflection equation with two eigenvalues. For example, one can take $\sum_{i=1}^{n} e^i_{n+1-i}$. Such matrices can be used for constructing solutions with three eigenvalues by the embedding method, along the line of Proposition 11.4. In this way, one gains the additional zero eigenvalue. One might have expected that this will provide a tool for quantizing all the semisimple orbits, which are classified as homogeneous spaces by the number of eigenvalues and their multiplicities. However, that is probably impossible, due to certain indications. Indeed, in [**KSS**], there were written out all non-degenerate solutions up to $n = 4$. They have at most two eigenvalues and lead to different quantizations for the symmetric orbits than by means of projectors. Among them, there are also q-traceless matrices. We conjecture that non-degenerate matrix solutions to the reflection equation with arbitrary value of the quantum trace do exist for each symmetric orbit. This implies the existence of the quantization via the RE algebra characters for every bisymmetric orbit, i.e., consisting of matrices with three eigenvalues. Let us present, following [**KSS**], the non-degenerate q-traceless RE matrices up to $n = 4$.

$$A^{1,1} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad A^{2,1} = \begin{pmatrix} -q^{-2} & & \hat{2}_q \\ & 1 & \\ 1 & & \end{pmatrix}$$

$$A^{2,2} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}, \quad A^{3,1} = \begin{pmatrix} -q^{-2}\hat{2}_q & & \hat{3}_q \\ & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}.$$

The matrix $A^{2,2}$ is interesting from the physical point of view because it yields the quantization of the twistor space. Using these RE matrices together with the projectors P_k, we can cover all the semisimple orbits for $n \leq 4$, excepting the maximal orbit in dimension $n = 4$. But the maximal orbits can be quantized by specifying values of the Casimir elements, i.e., $\mathrm{Tr}(L^i)$, $i = 1, \ldots, n-1$. So we conclude that, at least to dimension $n = 4$, all the semisimple orbits can be quantized explicitly as quotients of the reflection equation algebra.

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MAX-PLANCK-ISTITUTE FÜR MATHEMATIK, VIVATSGASSE 7, 53111 BONN
Current address: Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel
E-mail address: donin@macs.biu.ac.il

DEPARTMENT OF MATHEMATICS, BAR-ILAN UNIVERSITY, 52900 RAMAT-GAN, ISRAEL
E-mail address: mudrova@macs.biu.ac.il

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Toward a classification of Poisson structures on surfaces

Olga Radko

ABSTRACT. We consider topologically stable Poisson structures on a Riemann surface. These structures are defined as having the simplest (i.e., linear) degeneracies and form a dense set in the space of all Poisson structures on a given surface. We give a classification of such topologically stable structures by constructing a complete set of explicit invariants. We also explain how to extend these results to structures with higher order degeneracies; in the case of isolated quadratic degeneracies we construct a complete set of invariants.

1. Introduction and statement of main results

The aim of this paper is to initiate the classification of Poisson structures on simplest possible manifolds — surfaces — up to a Poisson isomorphism. Let Σ be a Riemann surface, and let ω be a non-degenerate 2-form on Σ. Then ω gives rise to a symplectic structure on Σ, which is by far the simplest kind of Poisson structure. Moser's theorem [**3**] in this case gives a complete classification of such symplectic structures:

THEOREM. *Let ω and ω' be two symplectic structures on Σ. Then there exists an orientation-preserving diffeomorphism $\varphi : \Sigma \to \Sigma$ so that $\varphi^*\omega = \omega'$ iff $\int_\Sigma \omega = \int_\Sigma \omega'$.*

Fix now a symplectic form ω on Σ, and denote by π the corresponding Poisson bivector. For any $f \in C^\infty(\Sigma)$, $f\pi$ is a Poisson bivector and hence $(\Sigma, f\pi)$ is a Poisson manifold. Moreover, any Poisson bivector can be written in such a form. Already locally the problem of classification of Poisson structures becomes non-trivial, in view of the different possibilities for the zero set of f and the order of its degeneracy on that set. Arnold [**1**] (see also Monnier [**2**]) obtained a hierarchical classification of germs of such Poisson structures, ordered by the type of degeneracy of f on its zero set. The simplest local situation is that of a locally nonzero f (in which case the structure $f\pi$ is symplectic and thus locally isomorphic to the one defined by π), followed by first-order degeneracy, in which case $f\pi$ is locally isomorphic to $g\pi$, where $g(x, y) = x$ in some local coordinates x, y. The next order of complexity involves quadratic (elliptic, $f = \lambda(x^2 + y^2)$, and hyperbolic, $f = \lambda(x^2 - y^2)$) degeneracies, and so on.

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The eventual goal of our work is to find a global classification of arbitrary structures $f\pi$ on a given surface. This goal presently being out of reach, we first turn to the task of classification of structures of the form $f\pi$ with the additional assumption that f has only linear degeneracies on Σ, i.e., that all of the degeneracies of $f\pi$ are first-order in the Arnold's hierarchy. The set of such structures is generic in the sense that it forms a dense subset in the linear space of all Poisson structures on Σ. We call the Poisson structures with at most linear degeneracies *topologically stable,* since the topology of their zero sets is preserved under small perturbations. Our hope to be able to classify at least the topologically stable structures was raised by the following result, in which we compute their Poisson cohomology:

THEOREM 1. *Given a topologically stable Poisson structure $f\pi$ on Σ, denote by $Z(f\pi)$ its zero set and let n be the number of its connected components (i.e., $Z(f\pi) = \bigsqcup_{i=1}^{n} \gamma_i$, where $\gamma_1, \ldots, \gamma_n$ are smooth simple closed non-intersecting curves). Then*

$$H^0_\pi(\Sigma, f\pi) \cong H^0(\Sigma),$$
$$H^1_\pi(\Sigma, f\pi) \cong \mathbb{R}^n \oplus H^1(\Sigma),$$
$$H^2_\pi(\Sigma, f\pi) \cong \mathbb{R}^n \oplus H^2(\Sigma) \cong \mathbb{R}^{n+1},$$

where H^*_π denotes the Poisson cohomology and H^* denotes the de Rham cohomology.

Note in particular that the second cohomology is finite-dimensional, predicting that there are only $n+1$ independent non-trivial deformations of the Poisson structure $f\pi$.

Indeed, we found that besides the obvious invariant of $f\pi$, namely the topology of the inclusion $Z(f\pi) \subset \Sigma$, we could further associate to $(\Sigma, f\pi)$ $n+1$ numeric invariants: the n *modular periods* of π around each connected component of $Z(f\pi)$ (Definition 1) and the *regularized Liouville volume* (Definition 2) which generalizes the total Liouville volume in the symplectic case. In addition, each closed smooth curve $\gamma \in Z(f\pi)$ received a preferred orientation (Remark 1). With these notations, we could then prove the following classification result:

THEOREM 2. *Given topologically stable Poisson structures $f\pi$ and $h\pi$ on a Riemann surface Σ, there exists an orientation-preserving diffeomorphism $\varphi : \Sigma \to \Sigma$ such that $\varphi_*(f\pi) = h\pi$ if and only if the following conditions are satisfied:*
(a) there exists an orientation-preserving diffeomorphism $\psi : \Sigma \to \Sigma$ so that $\psi(Z_f) = Z_h$ with ψ preserving the orientation of Z_f and Z_h ;
(b) for each connected component γ of Z_f, the modular period of $f\pi$ about γ is the same as the modular period of $h\pi$ around $\psi(\gamma)$;
(c) the regularized Liouville volumes of $f\pi$ and $h\pi$ are the same.

Although the theorem gives a classification of topologically stable Poisson structures on Σ up to orientation-preserving isomorphisms, one can remark that $f\pi$ and $h\pi$ are isomorphic in an orientation-reversing way iff for some orientation-reversing diffeomorphism $\theta : \Sigma \to \Sigma$ the structures $f\pi$ and $\theta_*(h\pi)$ are isomorphic in an orientation-preserving way. Whether or not this happens can be deduced from the classification theorem.

We also discuss the extension of the results above to Poisson structures $f\pi$, where in addition to linear degeneracies, f has a finite number of isolated quadratic

zeros. It turns out that the modular period and regularized volume invariants make sense in this greater generality; in addition, one gets an extra invariant for each quadratic zero of f. Once again, these invariants suffice to determine the structure $f\pi$ up to Poisson isomorphism. In principle, our approach can be applied to any of the higher-degeneracy Poisson structures on a Riemann surface.

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2. The invariants and the classification theorem

2.1. Modular periods. Let $(P, \{\,,\,\})$ be a Poisson manifold with the Poisson bracket $\{\,,\,\}$. Recall [6] that the *modular vector field* X^Ω of $(P, \{\,,\,\})$ with respect to a volume form Ω is defined by the equation

$$X^\Omega h \doteq \frac{L_{X_h}\Omega}{\Omega}, \quad h \in C^\infty(P),$$

where L_{X_h} is the Lie derivative with respect to $X_h \doteq \tilde{\pi}(dh)$, the Hamiltonian vector field of h. The flow of X^Ω turns out to preserve the Poisson structure of P. As is evident from its definition, X^Ω is zero iff $L_{X_h}\Omega = 0$ for all h, i.e., Ω is invariant under the flows of all Hamiltonian vector fields. If $\Omega' = h\Omega$ is another volume form, the difference

$$(X^{\Omega'} - X^\Omega) \cdot f = X_{\log h}$$

is a Hamiltonian vector field. In particular, if the Poisson tensor is zero on a set Z, the restriction $X^\Omega|_Z$ is independent of the choice of Ω, since $X_{\log h} f|_Z = 0$ for all h and f.

Returning now to our surface $(\Sigma, f\pi)$ with a topologically stable Poisson structure, consider a connected component $\gamma \subset Z(f\pi) = f^{-1}(\{0\})$. Since for any volume form Ω on Σ the modular vector field X^Ω preserves $f\pi$, it follows that the flow of X_Ω leaves the zero set $Z(f\pi)$ (and thus γ) globally invariant.

DEFINITION 1. (see also [5]) The *modular period* $T_\gamma(f\pi)$ of $f\pi$ around γ is defined to be the period of the flow of X^Ω around γ.

Since $X^\Omega|_\gamma$ is independent of the choice of Ω, the modular period is well-defined. The map $\gamma \mapsto T_\gamma(f\pi)$, $\gamma \in Z_f$ is an invariant of the Poisson structure $f\pi$.

Since $f\pi$ vanishes linearly on its zero set, it is given by the formula

$$f\pi(r,\theta) = cr\partial_r \wedge \partial_\theta, \quad c \neq 0$$

in an annular coordinate neighborhood $U \simeq S^1 \times (-1,1) = \{(r,\theta) | \, 0 \leq \theta \leq 2\pi, |r| < 1\}$ of a zero curve $\gamma = \{(r,\theta) | \, r = 0\} \subset Z(f\pi)$. In this case, $X^\Omega|_\gamma$ is given by the rotation with the period $T_\gamma(f\pi) = 2\pi/|c|$.

REMARK 1. Since the function f has different signs on the connected components of $\Sigma \setminus Z(f\pi)$ bounded by the same zero curve $\gamma \subset Z(f\pi)$, the modular vector field of $f\pi$ gives rise to an orientation on each curve $\gamma \subset Z(\pi)$.

REMARK 2. Similarly to the modular period around a closed zero curve, one can define the *modular flow time* along a segment of a zero curve on which the Poisson structure vanishes linearly.

2.2. Regularized Liouville volume. Recall that π is the Poisson structure associated to a fixed symplectic form ω. Thus on $\Sigma \setminus Z_f$ the Poisson structure $f\pi$ comes from the symplectic structure $\nu = \frac{1}{f}\omega$ (which is independent of the choice of π or, equivalently, ω).

DEFINITION 2. Define the *regularized Liouville volume* of $f\pi$ as the principal value of the integral

$$V(f\pi) = \text{P.V.} \int_{\Sigma \setminus Z(f\pi)} \nu$$

over the non-compact manifold $\Sigma \setminus Z_f$.

More precisely, one can show that for any function h vanishing linearly on $Z(f\pi)$ (i.e., such that $\frac{f}{h}$ is smooth on Σ), the limit

$$V(f\pi) = \lim_{\varepsilon \to 0} \int_{\{p \in \Sigma |\, |h(p)| > \varepsilon\}} \frac{1}{f}\omega$$

exists and does not depend on h. It is clear that $V(f\pi)$ is an invariant of the Poisson structure $f\pi$.

2.3. Sketch of the proof of Main Theorem. Let $f\pi$ and $g\pi$ be two topologically stable Poisson structures. Clearly, the coincidence of all of the invariants listed in Theorem 2 is necessary for $f\pi$ and $g\pi$ to be isomorphic. Conversely, assume that all the invariants are the same. Then $Z(f\pi) \subset \Sigma$ and $Z(g\pi) \subset \Sigma$ are diffeomorphic, so we may assume that $Z(f\pi) = Z(g\pi)$ by replacing $g\pi$ with an isomorphic structure. Since both f and g vanish linearly on each connected component $\gamma \subset Z(f\pi) = Z(g\pi)$, we can, by once again replacing $g\pi$ by an isomorphic structure, assume that in some annular coordinate neighborhood $U \simeq S^1 \times (-1, 1) = \{(r, \theta) |\, 0 \leq \theta \leq 2\pi,\ |r| < 1\}$,

$$\begin{aligned} f\pi &= cr\partial_r \wedge \partial_\theta, \quad c \neq 0, \\ g\pi &= dr\partial_r \wedge \partial_\theta, \quad d \neq 0. \end{aligned}$$

Since the modular periods of $f\pi$ and $g\pi$ are assumed to be the same, we have $2\pi/|c| = 2\pi/|d|$, which implies $|c| = |d|$. The fact that the orientations of $Z(f\pi)$ and $Z(g\pi)$ induced by modular vector fields are the same implies that $c = d$. Hence $f\pi$ and $g\pi$ are equal in a neighborhood of $Z(f\pi)$; by replacing $g\pi$ with an isomorphic structure we may assume that $f\pi = g\pi$ on a neighborhood U of $Z(f\pi)$.

Consider the non-compact manifold $N \doteq \Sigma \setminus Z(f\pi)$. The desired isomorphism of $f\pi$ and $g\pi$ on Σ would follow if we could argue that $f\pi$ and $g\pi$ are isomorphic on N via a diffeomorphism φ, so that φ is identity on $N \cap U$; indeed, φ could then be extended to a diffeomorphism $\bar{\varphi}$ of Σ by setting $\varphi(p) = p$ for $p \in Z(f\pi)$ and we would clearly have $\bar{\varphi}_*(f\pi) = g\pi$. Since $f\pi$ and $g\pi$ are nonzero on N, they come from symplectic structures $\frac{\omega}{f}$ and $\frac{\omega}{g}$ on N. These 2-forms coincide on $N \cap U$. Moreover, $\int_N (\frac{\omega}{f} - \frac{\omega}{g}) = V(f\pi) - V(g\pi) = 0$. Hence $\frac{\omega}{f}$ and $\frac{\omega}{g}$ define the same class in the compactly-supported de Rham cohomology of N. It is now a matter of imitating the proof of Moser's theorem to construct a time-dependent vector field supported on $N \setminus U$ and whose flow at time 1 carries $\frac{\omega}{f}$ into $\frac{\omega}{g}$. Taking φ to be that flow at time 1 we find that we have obtained the desired isomorphism.

For more details on the classification of topologically stable structures, see [**4**].

3. Structures with higher order degeneracies

Let $\gamma \subset S^2$ be a figure-eight curve with one self-intersection p; then there exists a Poisson structure $f\pi$ for which f vanishes on γ. However, one can no longer arrange that f vanishes linearly on its zero set. At the very least, f must vanish linearly on $\gamma \setminus \{p\}$ and around p have the expression $\lambda(x^2 - y^2)$ in some local coordinates (x, y).

More generally, we consider Poisson structures $f\pi$ on a surface Σ, for which f vanishes linearly on its zero set $Z(f\pi)$ except possibly at a number of points $p_1, \ldots, p_n \in Z(f\pi)$, where it could have quadratic zeros; i.e., $f(x, y) = \lambda_j(x^2 + \sigma_j y^2)$ for some $\lambda_j \neq 0$ and $\sigma_j \in \{\pm 1\}$ in a local coordinate system (x, y) around p_j. We call p_j, $j = 1, \ldots, n$ the *exceptional points* of the structure $f\pi$. If $\sigma_j = 1$ (resp., $\sigma_j = -1$) , the exceptional point p_j is called *elliptic* (resp., *hyperbolic*). Clearly, the exceptional points (as a subset of $Z(f\pi)$) are an invariant of $f\pi$.

Our aim is to extend the classification results of the previous section to such structures.

3.1. Classification at an exceptional point. Consider two Poisson structures $f\pi$ and $g\pi$ as above. When are these structures locally isomorphic in a neighborhood of an exceptional point?

LEMMA 1. *The numbers λ_j and σ_j are complete local invariants for a structure as above.*

More precisely, if $f = \lambda(x^2 + \sigma y^2)$ and $g = \lambda'(x^2 + \sigma' y^2)$ near a point p, then there exists a local Poisson isomorphism taking $f\pi$ to $g\pi$ if and only if $\lambda = \lambda'$ and $\sigma = \sigma'$.

The non-trivial part is to show that if such a local Poisson isomorphism exists, then $\lambda = \lambda'$. To do this, one notices that the modular vector field X of $f\pi$ vanishes at the point p and hence its derivative determines a linear operator on the tangent space $T_p\Sigma$. The spectrum of this operator is $\{\pm\lambda\}$ if $\sigma = -1$ and $\{\pm i\lambda\}$ if $\sigma = 1$, allowing one to recover λ from $f\pi$ in a coordinate-free way.

3.2. Local classification. Assume now that U and U' are two coordinate neighborhoods of an exceptional point p so that our Poisson structure is given by $f\pi$, where $f = \lambda(x^2 + \sigma y^2)$ on a neighborhood W near $p = (0, 0)$, and $U, U' \subset W$. When is $(U, f\pi|_U)$ Poisson isomorphic to $(U', f\pi|_{U'})$?

It turns out that the answers are different depending on whether $\sigma = 1$ or $\sigma = -1$.

3.2.1. *Elliptic case:* $\sigma = 1$. In this case, $(U, f\pi|_U)$ is always isomorphic to $(U, f\pi|_{U'})$. Indeed, let $D_r = \{(x, y) : x^2 + y^2 < r^2\}$ be the disk of radius r around the exceptional point p. Then there exist $r, r' > 0$, $D_r \subset U$, $D_{r'} \subset U'$ such that the symplectic volumes (with respect to $\frac{1}{f}\omega$) of $U \setminus D_r$ and $U' \setminus D_{r'}$ are equal. Using Moser's theorem, the question is reduced to whether $(D_r, f\pi|_{D_r})$ and $(D_{r'}, f\pi|_{D_{r'}})$ are isomorphic. The isomorphism of $(D_r, f\pi|_{D_r})$ and $(D_{r'}, f\pi|_{D_{r'}})$ is given by $(x, y) \mapsto \left(\frac{r'}{r}x, \frac{r'}{r}y\right)$.

3.2.2. *Hyperbolic case:* $\sigma = -1$. In this case, the situation is more intricate. The flow lines of a modular vector field X^Ω of $f\pi$ preserves the zero set $Z(f\pi) = \{(x, y) : x = \pm y\}$. As we remarked before, $X|_{Z(f\pi)}$ depends only on the Poisson structure $f\pi$ (and not on the choice of a volume form with respect to which it

is calculated). The subsets $Z^+(f\pi) = \{(x,y) : x = y\}$ and $Z^-(f\pi) = \{(x,y) : x = -y\}$ of $Z(f\pi)$ are canonically distinguished, since along one of them X points toward $(0,0)$, while along the other it points away from $(0,0)$.

Although the total flow time between the origin and any other point in $Z(f\pi)$ is infinite, it turns out that if $p = (a, \pm a)$ and $q = -(b, \pm b)$ are two points on $Z^{\pm}(f\pi)$ (same sign, $+$ or $-$, in all three expressions), then the *flow time* $I(p,q)$ of X from p to q can be assigned a finite number, as a principal value of the integral

$$I(p,q) \doteq \text{P.V.} \int_{-b}^{a} \frac{1}{x(t)} dt$$

(where $x(t)$ denotes the x-component of X at $(t, \pm t)$). As a result, we get two invariants of $(U, f\pi|_U)$:

$$I^{\pm}(U) = I^{\pm}(f\pi|_U) \doteq \sup_{p,q} I(p,q),$$

where the supremum is taken over all pairs of points $p, q \in Z^{\pm}(f\pi)$ such that p and q are to the opposite sides of $(0,0)$. These invariants reflect the maximal flow times of a modular vector field along the zero lines inside of U.

In addition to these invariants, note that the function f is positive on some parts of U and negative on the others. As a result, one can define a regularized volume invariant in the same spirit as before as the principal value of the integral

$$V(U) = V(f\pi|_U) = \text{P.V.} \int_U \frac{1}{f} \omega.$$

It turns out that the invariants $I^{\pm}(f\pi|_U)$ and $V(f\pi|_U)$ classify the structures $f\pi|_U$.

LEMMA 2. *With the notation as in the beginning of §3.2,* $(U, f\pi|_U) \cong (U', f\pi|_{U'})$ *if and only if* $V(f\pi|_U) = V(f\pi|_{U'})$ *and* $I^{\pm}(f\pi|_U) = I^{\pm}(f\pi|_{U'})$.

We sketch the proof. Let $W \in U \cap U'$ be an open neighborhood of $p = (0,0)$. Using the fact that $I^{\pm}(U) = I^{\pm}(U')$ one can find an isomorphism of the saturations $O \doteq \{q \in U : \exists t \text{ s.t. } \phi_t \in W\}$ and $O' \doteq \{q \in U' : \exists t \text{ s.t. } \phi_t \in W\}$ of W by the flow ϕ of the modular vector field X^{ω} (with respect to the usual volume form ω). Note that O (resp. O') is a neighborhood of $Z(f\pi) \cap U$ (resp., $Z(f\pi) \cap U'$). The assumption that $V(U) = V(U')$ guarantees that the Liouville volumes of the sets $U \setminus O$ and $U \setminus O'$ with respect to the form $\frac{1}{f}\omega$ are the same. Using Moser's theorem, one can now extend the isomorphism $(O, f\pi|_O) \cong (O', f\pi|_{O'})$ to an isomorphism of $(U, f\pi|_U)$ with $(U', f\pi|_{U'})$.

3.3. Global classification. Returning now to Poisson structures on Riemann surfaces, assume that π_1 and π_2 are Poisson structures on surfaces Σ_1 and Σ_2, so that π_α vanish linearly on their zero sets $Z(\pi_\alpha)$ except possibly at some points $p_\alpha^1, \ldots, p_\alpha^m \in \Sigma_\alpha$, $\alpha = 1, 2$. Assume furthermore that in a neighborhood U_α^j of p_α^j, π_α has the form $\pi_\alpha(x,y) = \lambda_\alpha^j(x^2 + \sigma_\alpha^j y^2)$, $\alpha = 1, 2$, where $\lambda_\alpha^j \neq 0$ and $\sigma_\alpha^j = \pm 1$.

Define the regularized Liouville volume invariant of π_α to be the sum

$$V(\pi_\alpha) \doteq \Sigma_{j=1}^m V\left(\pi_\alpha|_{U_\alpha^j}\right) + V\left(\pi_\alpha|_{\Sigma_\alpha \setminus (\bigsqcup_{j=1}^m U_\alpha^j)}\right),$$

where we set for convenience $V\left(\pi_\alpha|_{U_\alpha^j}\right) = \hat{\infty}$ (a special symbol) if $\sigma_\alpha^j = +1$, and we set $r + \hat{\infty} = \hat{\infty}$ if $r \in \mathbb{R} \cup \{\pm\infty\}$.

We now consider the zero set $Z(\pi_\alpha)$. Embed Σ_α into the three-dimensional space $\mathbb{R}^3$, and replace the zero set $Z(\pi_\alpha) \subset \Sigma_\alpha$ by a set $C(\pi_\alpha)$ of closed curves in $\mathbb{R}^3$ such that:

- $C(\pi_\alpha)|_{\Sigma_\alpha \setminus \bigsqcup_{j=1}^m U_\alpha^j} = Z(\pi_\alpha)|_{\Sigma_\alpha \setminus \bigsqcup_{j=1}^m U_\alpha^j}$;
- If p_α^j is an elliptic point ($\sigma_\alpha^j = 1$), then $C(\pi_\alpha)|_{U_\alpha^j} = Z(\pi_\alpha)|_{U_\alpha^j}$;
- If p_α^j is a hyperbolic point ($\sigma_\alpha^j = -1$), then $C(\pi_\alpha)|_{U_\alpha^j}$ consists of two non-intersecting curves obtained by resolving the intersection of the sets Z^+ and Z^- (see §3.2.2 for the notation).

Assign to each resulting closed curve $\gamma \subset C(\pi_\alpha)$ its modular period $I_\gamma(\pi)$ by adding the modular flow times between the exceptional points on γ.

THEOREM 3. *With the above notation and assumptions, let $\phi : \Sigma_1 \to \Sigma_2$ be a diffeomorphism, so that $\phi(Z(\pi_1)) = Z(\pi_2)$ and $\phi(p_1^i) = \phi(p_2^i)$, $i = 1, \dots, m$. Assume that the map $Z(\pi_1) \to Z(\pi_2)$ induces the map $\phi^C : C(\pi_1) \to C(\pi_2)$. Then there exists a Poisson isomorphism $\theta : \Sigma_1 \to \Sigma_2$, $\theta_* \pi_1 = \pi_2$ such that $\theta|_{Z(\pi_1)} = \phi^C$ if and only if*

(1) $I_\gamma(\pi_1) = I_{\phi^C(\gamma)}(\pi_2)$ *for all* $\gamma \in C(\pi_1)$;
(2) $V(\pi_1) = V(\pi_2)$;
(3) $\lambda_1^i = \lambda_2^i$, $\sigma_1^i = \sigma_2^i$ *for all* $i = 1, \dots, m$;

We sketch the proof.

Clearly, $\gamma \mapsto I_\gamma(\pi_k)$ and $V(\pi_k)$ are invariants, and hence the conditions are necessary.

Conversely, assume that the invariants are the same. Choose a curve $\gamma \in C(\pi_1)$. Choose points a_1^i, b_1^i on γ so that $p_1^i \in [\alpha_1^i, \beta_1^i] \subset \gamma$ for all exceptional points p_1^i on γ. Do the same for the corresponding curve in $C(\pi_2)$, calling the points a_2^i, b_2^i. The assumption (1) guarantees that this can be done in such a way so that $I(a_1^i, b_1^i) = I(a_2^i, b_2^i)$ and $T_{[a_1^i, b_2^{i+1}]}(\pi_1) = T_{[a_2^i, b_2^{i+1}]}(\pi_2)$. Now choose disjoint contractable neighborhoods $U_\alpha^i \subset \Sigma_\alpha$ of p_α^i so that $V(U_1^i) = V(U_2^i)$ for all i. Using the results of the previous section, we can now find Poisson isomorphisms of U_1^i and U_2^i for all i. Similar procedure should be done for all the curves in $C(\pi_1)$, $C(\pi_2)$. The assumption (2) guarantees that $V(\Sigma_1 \setminus (U_1^1 \cup \dots \cup U_1^m)) = V(\Sigma_2 \setminus (U_2^1 \cup \dots \cup U_2^m))$. One then argues as in Moser's theorem that these isomorphisms can be extended to the desired global isomorphism ϕ.

3.4. Remarks. It should be noted that a Poisson structure $f\pi$ with quadratic degeneracies can be always resolved to a Poisson structure with only linear degeneracies by replacing f with $f + \varepsilon 1$ (here 1 denotes the constant function 1). It is possible (although more tedious) to recapture the invariants λ_j, I_γ and V by considering invariants of the structures $f + \varepsilon 1$. One advantage of this point of view is the explanation of the disappearance of the volume invariant for structures of the form $(x^2 + y^2)\pi$ (restricted to some neighborhood of zero U). Indeed, if we denote by V_ε the volume invariant of the structure $(x^2 + y^2 + \varepsilon)\pi|_U$, we have $V(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

We remark that in the proof above, only §3.1 and §3.2 used the fact that $f\pi$ had quadratic degeneracies at the special points. One can in principle try to carry out a similar argument for structures with higher-order singularities (such as for example the structures appearing in Arnold's hierarchy given in [1]); the essential part is to obtain the analogs of §3.1 and §3.2 for the desired class of structures.

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Department of Mathematics, UC Berkeley, Berkeley, CA 94720
E-mail address: radko@math.berkeley.edu

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Lectures on operads

J. D. S. Jones

ABSTRACT. These notes are based on a series of introductory lectures on the theory of operads given at the conference held in UMIST in July 2001.

1. Introduction

This is a written version of a series of lectures given at the conference on quantization, deformations, and new homological an categorical methods in mathematical physics, held in UMIST in July 2001. The main theme is to discuss how operads are used to formalize and study the notion of a "structure up to homotopy" and give applications of this theory in geometry (in the broadest sense), quantization, and homological algebra.

Stasheff's theory of A_∞ spaces and A_∞ algebras illustrates many of the key ideas. An A_∞ space is a topological space equipped with a product operation which is associative up to homotopy and the homotopy which makes the product associative can be chosen so that it satisfies a collection of higher coherence conditions. These coherence conditions involve homotopies between homotopies and are most neatly formulated in terms of the so-called Stasheff polyhedron (or asssociahedron). The prime examples of A_∞ spaces are based loop spaces and Stasheff proved that a connected A_∞ space is indeed homotopy equivalent to a loop space. An A_∞ algebra is a chain complex equipped with a product which is homotopy associative and the homotopy which makes the product associative can be chosen so that it satisfies suitable algebraic higher coherence conditions (in the context of chain complexes, homotopy is to be interpreted as chain homotopy). The link between the two notions is given by the combinatorial structure of the Stasheff polyhedra. In many ways, this is the model situation, there is a geometric construction, an algebraic construction and the two are linked by combinatorializing the geometrical construction.

Topological operads were introduced by May in his monograph [**15**]. The main reason for studying topological operads is to generalize Stasheff's theory of loop spaces to n-fold loop spaces and infinite loop spaces – operads play the role of the Stasheff polyhedra in this more general theory.

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Recently there has been a lot of interest in the algebraic theory of operads, much of it stimulated by theoretical physics. A particularly striking example is Kontsevich's formality theorem in deformation quantization [**12**]. An essential ingredient in this theorem is the notion of a Lie algebra up to homotopy (or a Gerstenhaber algebra up to homotopy) and this is where algebraic operads come in to the picture. The relevant operads are examples of Koszul operads and the theory of Koszul operads has many very interesting special features.

The operads which play a role in deformation quantization are also closely related to configuration spaces, in particular the Fulton–MacPherson compactification of the configuration spaces of k points in $\mathbb{R}^n$. If $n = 1$ these Fulton–MacPherson compactifications are the same as the Stasheff polyhedra. From the theory of topological operads we know that the configuration spaces of points in $\mathbb{R}^n$ are related to n-fold loop spaces. In many ways, these Fulton–MacPherson compactifications provide the most natural generalizations of the Stasheff polyhedra.

Here we will describe the algebraic aspects of the above ideas. There was not enough time in the lectures to do justice to the geometric part of the theory, that is to make a proper study of the Fulton–MacPherson compactification of the configuration spaces and the link between these spaces and the algebraic operads which arise in deformation quantization. Neither was there enough time, nor space, to include this in the written account of these lectures, however see [**7**] for an account of some of this material.

I have concentrated, here, on trying to explain the main lines in the theory. In particular I have left out many details and I have not tried to be systematic, and historically accurate, in the references. However, the recent book [**14**] gives a very thorough account of many of these missing details and it contains a very good section on the historical development of the theory of operads and an extensive bibliography.

There are two people I would particularly like to thank for their help in preparing this written version of the lectures given at the conference. The first is Ted Voronov for encouraging me to write up these lectures and for his patience as I did so. The second is John Rawnsley whose knowledge of deformation quantization and interest in Kontsevich's formality theorem have been an enormous help to me.

2. Operads: the basic definitions and examples

2.1. Preliminaries. The idea behind the definition of an operad is this. An algebraic structure on a vector space V can be described by giving all the multilinear operations in the elements of V this structure allows us to form, the symmetries of these operations, and the relations between them. We now formalize this idea.

Let V be a vector space over a fixed ground field K. Then, for each integer $k \geq 0$, define

$$\mathcal{E}_V(k) = \mathrm{Hom}(V^{\otimes k}, V).$$

So, $\mathcal{E}_V(k)$ is the space of all k-linear operations on V. Since $V^{\otimes 0} = K$ it follows that $\mathcal{E}_V(0) = V$. The symmetric group Σ_k acts on $\mathcal{E}_V(k)$ by permuting the factors in $V^{\otimes k}$ (by convention, Σ_0 is the trivial group) and there are substitution operations

$$\mathcal{E}_V(k) \otimes \mathcal{E}_V(n_1) \otimes \cdots \otimes \mathcal{E}_V(n_k) \to \mathcal{E}_V(n), \quad n = n_1 + \cdots + n_k$$

defined by

$$(\phi \otimes \psi_1 \otimes \cdots \otimes \psi_k) \mapsto \phi \circ (\psi_1 \otimes \cdots \otimes \psi_k).$$

There are relations between these operations.

(1) The *substitution relations*, arise when we replace the ψ_i in the substitution operations by

$$\phi_{j_i} \circ (\psi_{i,1} \otimes \cdots \otimes \psi_{i,j_i}).$$

(2) The *equivariance conditions*, express the interaction between the action of Σ_j on $\mathcal{E}_V(j)$ and the substitution operations.

We will now describe these relations in detail using order preserving maps between finite ordered sets and the interaction between permutations and order preserving maps of finite sets.

The substitution operations are naturally parametrized by ordered partitions. An *ordered partition* of n of length k is an ordered set of integers $\{n_1, \ldots, n_k\}$ such that $n_i \geq 0$ for $1 \leq i \leq k$ and $n = n_1 + \cdots + n_k$. It is important to note that some of the n_i are allowed to be zero. The substitution relations arise when we start with a partition $\{n_1, \ldots, n_k\}$ and then partition each of the integers n_i, i.e. they arise from partitions of partitions. Partitions are the same as order preserving maps between finite ordered sets and 'partitions of partitions' correspond to composition of order preserving maps.

More precisely, for each integer n with $n \geq 1$, let $\sigma(n)$ be the set $\{1, \ldots, n\}$ with its natural ordering. Let $\sigma(0)$ to be the empty set. Let $f : \sigma(n) \to \sigma(k)$ be an order preserving map. Then we get an ordered partition $P = P_f$ of n by setting $n_i = |f^{-1}(i)|$. Clearly the correspondence $f \mapsto P_f$ defines a bijection between the set of order preserving maps $\sigma(n) \to \sigma(k)$ and the set of ordered partitions of n of length k. Let Δ be the category with objects the ordered sets $\sigma(n)$, where $n \geq 0$, and morphisms the order preserving maps between these sets.

In the category Δ there is an operation, *disjoint union* $\sqcup$, on morphisms. This arises from the fact that an ordered disjoint union of ordered sets has a natural ordering. Therefore there is a canonical identification of $\sigma(n) \sqcup \sigma(m)$ with $\sigma(n+m)$. Given order preserving maps $f : \sigma(m) \to \sigma(j)$ and $g : \sigma(n) \to \sigma(k)$ then these identifications define an order preserving map

$$f \sqcup g : \sigma(m+n) \to \sigma(j+k).$$

Explicitly $f \sqcup g$ is defined as follows:

$$f \sqcup g(s) = \begin{cases} f(s) & \text{if } 1 \leq s \leq m \\ g(s) + j & \text{if } m+1 \leq s \leq m+n \end{cases}$$

Furthermore, if $f : \sigma(n) \to \sigma(k)$ is an order preserving map then there is a unique decomposition of f in the form

$$f = f_1 \sqcup \cdots \sqcup f_k$$

where $f_i : \sigma(|f^{-1}(i)|) \to \sigma(1)$. The fact that we have defined $\sigma(0)$ to be the empty set allows us to deal with the possibility that $f^{-1}(i)$ is empty.

Let $f : \sigma(n) \to \sigma(k)$ be an order preserving map and let $\pi \in \Sigma_k$ be a permutation. Let $S_i \subset \sigma(n)$ be the set $f^{-1}(i)$. Then we get a permutation $f^*(\pi) \in \Sigma_n$ by permuting the sets S_i according to π. We also get an order preserving map $\pi_*(f) : \sigma(n) \to \sigma(k)$; this is the order preserving map corresponding to the partition

$$n = |S_{\pi(1)}| + \cdots + |S_{\pi(k)}|.$$

The analogous interaction between order preserving maps and cyclic permutations appears in Connes's theory of cyclic homology where it is formalized in Connes's category Λ, see for example [**3**] and also [**10**]. Fiedorowicz and Loday, see [**4**] generalized Connes's category using the idea of a crossed simplicial module. In the notation of [**4**] we are using the category $\Delta\Sigma$ in the above definition of an operad. We give the precise definition of this category.

The category $\Delta\Sigma$ is defined as follows. There is one object $\sigma(n)$ for each integer $n \geq 0$. The set of morphisms in $\Delta\Sigma$ from $\sigma(n) \to \sigma(k)$ is denoted by $\Delta\Sigma(n,k)$ and is given by

$$\Delta\Sigma(n,k) = \Delta(n,k) \times \Sigma(n)$$

where $\Delta(n,k)$ be the set of order preserving maps $\sigma(n) \to \sigma(k)$. The composition law is defined by the formula

$$(s,\sigma) \circ (t,\tau) = (s \circ (\sigma_* t), (t^*\sigma) \circ \tau).$$

Here $s \in \Delta(n,k)$, $\sigma \in \Sigma_n$, $t \in \Delta(m,n)$, $\tau \in \Sigma_m$; the operations $\sigma_* t$ and $t^*\sigma$ are those described above; and the first symbol $\circ$ refers to composition of order preserving maps and the second is composition of permutations.

The role of this category $\Delta\Sigma$ in the study of operads and the applications of operads will be pursued elsewhere. However, one of the main themes in these notes is to make systematic use of order preserving maps between finite ordered sets and the interaction between such maps and permutations when describing the basic theory of operads. This point of view is very close, but not absolutely identical, to the point of view on the definition of an operad taken in [**14**]

2.2. The definition of an operad. Suppose now we are given a sequence of vector spaces $\mathfrak{a}(k)$ where $k \geq 0$, and that $\mathfrak{a}(k)$ is a representation of the symmetric group Σ_k. Let $f : \sigma(n) \to \sigma(k)$ be an order preserving map and let $n_i = |f^{-1}(i)|$. Define $\mathfrak{a}(f)$ to be the vector space

$$\mathfrak{a}(f) = \mathfrak{a}(n_1) \otimes \cdots \otimes \mathfrak{a}(n_k).$$

Note that $\mathfrak{a}(f \sqcup g) = \mathfrak{a}(f) \otimes \mathfrak{a}(g)$. Define $\Sigma(f)$ to be the group

$$\Sigma(f) = \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}.$$

Then $\mathfrak{a}(f)$ is a representation of $\Sigma(f)$. If $\pi \in \Sigma_k$ is a permutation we get an isomorphism

$$\Sigma(\pi) : \Sigma(f) \to \Sigma(\pi_* f) \quad \mathfrak{a}(\pi) : \mathfrak{a}(f) \to \mathfrak{a}(\pi_* f)$$

and $\mathfrak{a}(\pi)$ is equivariant with respect to the isomorphism of groups $\Sigma(\pi)$–in formulas

$$\mathfrak{a}(\pi)(\theta x) = \Sigma(\pi)(\theta)\mathfrak{a}(\pi)(x)$$

for $x \in \mathfrak{a}(f)$ and $\theta \in \Sigma(f)$.

Suppose, in addition, we are given linear maps, which will usually be referred to as the *structure maps*,

$$\mathfrak{a}(k) \otimes \mathfrak{a}(n_1) \otimes \cdots \otimes \mathfrak{a}(n_k) \to \mathfrak{a}(n) \quad n = n_1 + \cdots + n_k.$$

These linear maps are naturally parametrized by the morphisms in Δ. If $f : \sigma(n) \to \sigma(k)$ is an order preserving map, then there is a unique structure map

$$\mu_f : \mathfrak{a}(k) \otimes \mathfrak{a}(f) \to \mathfrak{a}(n).$$

Now if

$$f : \sigma(n) \to \sigma(k), \quad g : \sigma(k) \to \sigma(j)$$

are order preserving maps we get a linear map

$$\mathfrak{a}(g) \otimes \mathfrak{a}(f) \to \mathfrak{a}(g \circ f).$$

This is constructed as follows. Since every morphism in Δ is a disjoint union of morphisms with target $\sigma(1)$ and $\mathfrak{a}$ converts disjoint unions into tensor products, it is sufficient to deal with the case where $j = 1$. In this case

$$\mathfrak{a}(g) = \mathfrak{a}(k), \quad \mathfrak{a}(g \circ f) = \mathfrak{a}(n)$$

$$\mathfrak{a}(f) = \mathfrak{a}(n_1) \otimes \cdots \otimes \mathfrak{a}(n_k),$$

where $n_i = |f^{-1}(i)|$. The structure map $\mathfrak{a}(k) \otimes \mathfrak{a}(n_1) \otimes \cdots \otimes \mathfrak{a}(n_k) \to \mathfrak{a}(n)$ is exactly the required map $\mathfrak{a}(g) \otimes \mathfrak{a}(f) \to \mathfrak{a}(g \circ f)$.

Note that the structure maps themselves are maps of the form $\mathfrak{a}(g) \otimes \mathfrak{a}(f) \to \mathfrak{a}(g \circ f)$. If $f : \sigma(n) \to \sigma(k)$ is an order preserving map take $g : \sigma(k) \to \sigma(1)$ to be the constant map and then, in this case, the map $\mathfrak{a}(g) \otimes \mathfrak{a}(f) \to \mathfrak{a}(g \circ f)$ is the structure map μ_f.

2.2.1. *The substitution relations.* The substitution relations are the following associativity conditions. Given order preserving maps

$$f : \sigma(m) \to \sigma(n), \quad g : \sigma(n) \to \sigma(k), \quad h : \sigma(k) \to \sigma(j)$$

the following diagram commutes

$$\begin{array}{ccc} \mathfrak{a}(h) \otimes \mathfrak{a}(g) \otimes \mathfrak{a}(f) & \longrightarrow & \mathfrak{a}(h \circ g) \otimes \mathfrak{a}(f) \\ \downarrow & & \downarrow \\ \mathfrak{a}(h) \otimes \mathfrak{a}(g \circ f) & \longrightarrow & \mathfrak{a}(h \circ g \circ f) \end{array}$$

It is a good exercise to take the case where $j = 1$ and write out this commutative diagram in detail to get the commutative diagram in Definition 1.1 (a) of [**11**].

2.2.2. *The equivariance conditions.* There are two equivariance conditions.

(1) For each order preserving map $f : \sigma(n) \to \sigma(k)$ the structure map

$$\mu_f : \mathfrak{a}(k) \otimes \mathfrak{a}(f) \to \mathfrak{a}(n)$$

is equivariant with respect to Σ_f. This group acts on $\mathfrak{a}(k) \otimes \mathfrak{a}(f)$ by extending its action on $\mathfrak{a}(f)$ by the trivial action on $\mathfrak{a}(k)$ and it acts on $\mathfrak{a}(n)$ as a subgroup of Σ_n.

(2) For any order preserving map $f : \sigma(n) \to \sigma(k)$ and permutation $\pi \in \Sigma_k$ the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{a}(k) \otimes \mathfrak{a}(f) & \xrightarrow{\pi \otimes \mathfrak{a}(\pi)} & \mathfrak{a}(k) \otimes \mathfrak{a}(\pi_* f) \\ {\scriptstyle \mu_f}\downarrow & & \downarrow{\scriptstyle \mu_{\pi_* f}} \\ \mathfrak{a}(n) & \xrightarrow[\mathfrak{a}(f_* \pi)]{} & \mathfrak{a}(n). \end{array}$$

Again it is a good exercise to compare this diagram with the corresponding diagram in [**11**].

DEFINITION 2.1. An operad $\mathfrak{a}$ is a sequence of vector spaces $\mathfrak{a}(k)$ for $k \geq 0$ such that $\mathfrak{a}(k)$ is a representation of the symmetric group Σ_k, equipped with structure maps

$$\mu_f : \mathfrak{a}(k) \otimes \mathfrak{a}(f) \to \mathfrak{a}(k),$$

defined for each order preserving map $f : \sigma(n) \to \sigma(k)$, satisfying the substitution relations and the equivariance conditions.

A *morphism of operads* $\mathfrak{a} \to \mathfrak{b}$ is a sequence of equivariant linear maps $\mathfrak{a}(n) \to \mathfrak{b}(n)$ preserving the structure maps in the obvious way. Clearly, with this definition of morphism, operads form a category.

We will usually want an operad to have a unit. This condition can be formulated as follows. Define

$$I(k) = \begin{cases} K & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then I is an operad in an obvious way. If $\mathfrak{a}$ is an operad a unit for $\mathfrak{a}$ is a map of operads $\eta : I \to \mathfrak{a}$. In terms of elements this means that there is a preferred element $e \in \mathfrak{a}(1)$ such that $\mu(e \otimes x) = x$ and $\mu(y \otimes e \otimes \cdots \otimes e) = y$. Here the notation is meant to be obvious: if it is not, then it is a good exercise to work out what it should be.

2.3. Algebras over an operad. The spaces $\mathcal{E}_V(k)$, defined in §2.1, form an operad. This is the prototypical example of an operad, the *endomorphism operad* $\mathcal{E}_V$ of V and the definition of an operad is given by abstracting the structure present in this example. There are obvious evaluation maps

$$\mathcal{E}_V(k) \otimes V^{\otimes k} \to V$$

which satisfy universal relations. This is the prototypical example of an algebra over an operad. We now abstract the structure present in this example to give the definition of an algebra over an operad.

A vector space V is an *algebra over the operad* $\mathfrak{a}$ if we are given a map of operads $\mathfrak{a} \to \mathcal{E}_V$. More explicitly a vector space V is an algebra over $\mathfrak{a}$ if there are structure maps

$$\mathfrak{a}(k) \otimes V^{\otimes k} \to V$$

which are equivariant and satisfy appropriate substitution relations. In the equivariance condition the group Σ_k acts on $\mathfrak{a}(k) \otimes V^{\otimes k}$ by the tensor product of the given action on $\mathfrak{a}(k)$ and the permutation action on $V^{\otimes k}$ and it acts trivially on the target V of the structure maps. The substitution relations are given by a straightforward adaptation of the corresponding relations for an operad and we will not go into detail. There is also the natural notion of a unital algebra over a unital operad; again we will not go into detail.

If V is an algebra over the operad $\mathfrak{a}$ then the space $\mathfrak{a}(k)$ exactly parametrizes the k-linear operations it is possible to form in V using the given algebraic structure, the symmetries of these operations are encoded in the action of the symmetric groups and the relations between them are given by the structure maps.

2.4. Examples of operads. The first example of an operad is the endomorphism operad $\mathcal{E}_V$ of a vector space V. Here is another simple example. If A is an algebra over the ground field K then define

$$I_A(k) = \begin{cases} A & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then I_A is an operad: there is one non-trivial structure map $I_A(1) \otimes I_A(1) \to I_A(1)$ and this is defined by the product in A; the equivariance condition is vacuous; there is one substitution relation and this is the associativity of the product in A. This operad is unital if and only if the algebra is unital. Algebras over the operad I_A are A modules.

2.4.1. *The associative algebra operad.* Consider the sequence of representations

$$\mathfrak{ass}(k) = K(\Sigma_k)$$

where $K(\Sigma_k)$ is the regular representation of the symmetric group. The structure maps are defined as follows. Let $f : \sigma(n) \to \sigma(k)$ be an order preserving map. Then we must give a linear map

$$\mathfrak{ass}(k) \otimes \mathfrak{ass}(f) \to \mathfrak{ass}(n).$$

Now a basis for $\mathfrak{ass}(k) \otimes \mathfrak{ass}(f)$ is given by the set

$$\Sigma_k \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$$

where $n_i = |f^{-1}(i)|$ and the structure map is defined by mapping the basis element

$$(\sigma, \pi_1, \ldots, \pi_k)$$

to the permutation

$$f^*(\sigma) \circ (\pi_1 \sqcup \cdots \sqcup \pi_k)$$

where $\sqcup$ is the disjoint union (or block sum) operation on permutations. It is straightforward to check that these maps define an operad structure on the spaces $\mathfrak{ass}(k)$

An algebra over $\mathfrak{ass}$ is just an associative algebra. This corresponds to the fact that a basis for the k-linear operations it is possible to form in an associative algebra is A given by the set of operations

$$a_1 \otimes \cdots \otimes a_k \mapsto a_{\pi(1)} \cdots a_{\pi(k)}$$

where $\pi \in \Sigma_k$.

2.4.2. *The commutative algebra operad.* Now consider the sequence of representations

$$\mathfrak{com}(k) = K$$

where K is the trivial one dimensional representation of Σ_k. These spaces form an operad, $\mathfrak{com}$. Of course, the operad structure is rather trivial. Algebras over $\mathfrak{com}$ are just commutative algebras since, up to multiplication by scalars, there is only one k-linear operation it is possible to form in a commutative algebra, the operation

$$a_1 \otimes \cdots \otimes a_k \mapsto a_1 \cdots a_k.$$

Since the algebra is commutative, the order in which the elements are multiplied does not matter.

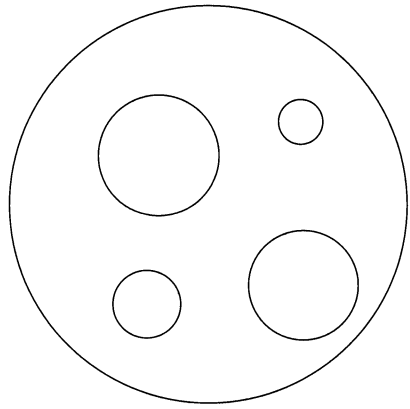
2.5. Linear operads, graded operads, differential graded operads and topological operads. From now on, we use the term linear operads for operads in the category of vector spaces, that is operads as defined in §1.2. Clearly, we should consider algebraic structures on graded vector spaces (the term graded, unqualified, will mean $\mathbb{Z}$ graded) and so we will need the notion of a *graded operad.* This simply means we assume the representations are graded, use the graded tensor product, ask that the structure maps preserve grading and use the usual sign conventions of graded algebra when permuting terms in a graded tensor product. We can also define *differential graded operads* by replacing vector spaces by chain complexes, using the tensor product of chain complexes and asking that the structure maps are maps of chain complexes. Furthermore, we can define *topological operads* by replacing vector spaces by topological spaces, tensor product by cartesian product and asking that the structure maps are continuous. Indeed we can define operads in any symmetric monoidal category, this is a category equipped with an operation of two variables (like tensor product of vector spaces or cartesian product of topological spaces) which is associative and commutative in an appropriate sense, see [**13**].

Here we will be mainly interested in linear operads, graded operads, differential graded operads, and topological operads. Notice that there are natural maps

$$\text{topological operads} \to \text{differential graded operads} \to \text{graded operads}$$

given by taking singular chains, and homology respectively. This sequence of arrows is very important in the study of operads. Indeed, to be provocative, I do not think that, at present, there is an interesting graded operad which does not arise in some way from a topological operad.

2.6. The little d-discs operad. We now give an example of a topological operad. Let $D^d(a,r)$ be the closed disc in $\mathbb{R}^d$ with centre a and radius r and let D^d be the disc $D^d(0,1)$, that is the standard unit d-disc. Now let $\mathcal{D}_d(k)$ be the space of k ordered disjoint d discs in D^d.

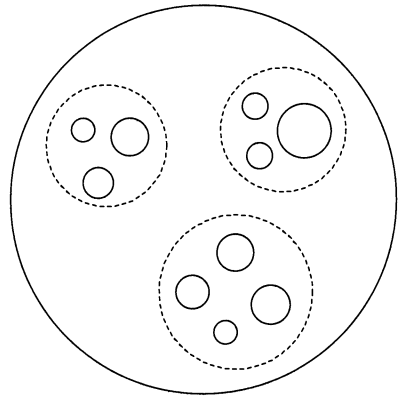


Thus a point of $\mathcal{D}_d(k)$ is parametrized by k points $a_1,\dots,a_k$ in the interior of D^d, (the centres of the little discs), and k strictly positive real numbers $r_1,\dots r_k$, (the radii of the little discs). The centres $a_1,\dots a_k$ and the radii $r_1,\dots r_k$ must satisfy some conditions in order that the corresponding discs lie in D^d and are disjoint. Taking centres and radii gives an injective function

$$\mathcal{D}_d(k) \to \mathbb{R}^{k(d+1)}$$

and using this function to identify $\mathcal{D}_d(k)$ with a subset of $\mathbb{R}^{k(d+1)}$ we give $\mathcal{D}_d(k)$ the subspace topology.

The operad structure of this sequence of topological spaces is given as follows. The symmetric group acts on $\mathcal{D}_d(k)$ by permuting the discs. The structure maps are given by the 'discs in discs in discs' operations.



More precisely, given a disc with centre a and radius r contained in D^d we get an embedding

$$i_{a,r} : D^d \to D^d, \quad x \mapsto a + |x| r x$$

with image $D^d(a,r)$. Using this embedding we can identify a configuration of discs in D^d with a configuration of discs in $D^d(a,r)$. Now we use this construction in the natural way to get maps

$$\mathcal{D}_d(k) \times \mathcal{D}_d(n_1) \times \cdots \times \mathcal{D}_d(n_k) \to \mathcal{D}_d(n)$$

where, as usual, $n = n_1 + \cdots + n_k$. These structure maps make the sequence of spaces $\mathcal{D}_d(k)$ into a topological operad, the little d-discs operad.

We now consider algebras over the little discs operad. A space X is an algebra over the operad $\mathcal{D}_d$ if we are given maps

$$\mathcal{D}_d(k) \times X^k \to X$$

satisfying certain conditions. Let Y be a topological space with base-point y_0. Let $\Omega^d Y$ be the d-fold loop space of Y which will be identified with the space

$$\mathrm{Map}((D^d, S^{d-1}), (Y, y_0))$$

of maps of D^d to Y which map the boundary S^{d-1} to the base-point y_0. The space $\Omega^d Y$ is an algebra over the operad $\mathcal{D}_d$. To see this, suppose we have k disjoint discs

$$D(a_j, r_j) \subset D^d, \quad 1 \leq j \leq k$$

and k maps

$$f_j : (D^d, S^{d-1}) \to (Y, y_0), \quad 1 \leq j \leq k.$$

Then there is a unique map

$$f : (D^d, S^{d-1}) \to (Y, y_0)$$

which is constant outside the union of the little discs $D(a_j, r_j)$ and such that

$$i_{a_j, r_j} \circ f = f_j.$$

where $i_{a_j, r_j} : D^d \to D^d$ is the canonical embedding of D^d with image $D(a_j, r_j)$.

These maps make $\Omega^d Y$ into an algebra over the little d-discs operad. Indeed the following theorem shows that, up to homotopy, these are essentially the only examples of algebras over the little d-discs operad.

THEOREM 2.2. *Suppose X is a connected space which is an algebra over the topological operad $\mathcal{D}_d$. Then there is a space Y such that*

$$X \simeq \Omega^d(Y).$$

For more information on this theorem and other theorems of this type see [**15**] where operads were first systematically used. Here are some observations on this theorem.

(1) There is a version of this theorem if the space X is not connected — it involves the notion of group completion, see [**15**] for the details.
(2) Theorems of this kind are the origins of operads. It is important to note how this theorem can be informally paraphrased by saying that a space X is a d-fold loop space up to homotopy if and only if it is an algebra over the operad $\mathcal{D}_d$. This makes the point that operads arose from the problem of trying to give a precise way of describing structures up to homotopy.
(3) The operad $\mathcal{D}_2$ has something to do with conformal/topological field theory as described by Graeme Segal. For example, suppose a space X is an algebra over the operad $\mathcal{D}_2$. Then a pair of pants (i.e. configuration of two discs in a disc) gives a product

$$X \times X \to X.$$

This is because there is a map

$$\mathcal{D}_2(2) \times X \times X \to X$$

and a pair of pants is a point in $\mathcal{D}_2(2)$.

2.7. Operads derived from the homology of configuration spaces. Now we look at the algebraic operads $H_*(\mathcal{D}_d)$ defined, in the obvious way, by

$$H_*(\mathcal{D}_d)(k) = H_*(\mathcal{D}_d(k)).$$

The space $\mathcal{D}_d(k)$ has the homotopy type of the configuration space $C_k(\mathbb{R}^d)$ consisting of k ordered distinct points in $\mathbb{R}^d$. Taking the centres of the little discs gives k ordered distinct points in the open unit disc in $\mathbb{R}^d$ and combining this with the usual homeomorphism of the open disc with $\mathbb{R}^d$ gives a map

$$\mathcal{D}_d(k) \to C_k(\mathbb{R}^d).$$

A simple argument shows that this map is a homotopy equivalence. Now we give the theorem computing the homology and cohomology of the spaces $C_k(\mathbb{R}^d)$

If $1 \leq i, j \leq k$ and $i \neq j$ there are maps

$$p_{i,j} : C_k(\mathbb{R}^d) \to S^{d-1}$$

defined by

$$p_{i,j}(x_1, \ldots, x_k) = \frac{x_i - x_j}{|x_i - x_j|}$$

By pulling back the fundamental cohomology class of S^{d-1} we get cohomology classes

$$\omega_{i,j} \in H^{d-1}(C_k(\mathbb{R}^d)), \quad 1 \leq i, j \leq k,\ i \neq j.$$

THEOREM 2.3. *The cohomology ring $H^*(C_k(\mathbb{R}^d))$ is the graded commutative ring with generators $\omega_{i,j}$ and relations*

$$\omega_{i,j} = (-1)^d \omega_{j,i}$$

$$\omega_{i,j}\omega_{j,k} + \omega_{j,k}\omega_{k,i} + \omega_{k,i}\omega_{i,j} = 0$$

$$\omega_{i,j}^2 = 0 \quad \textit{if } d \textit{ is odd.}$$

This theorem is proved in many places, see for example [**1, 2**]. Since the groups $H^*(C_k(\mathbb{R}^d))$ are free abelian groups it follows that the homology groups are completely determined by this theorem.

Now define the graded operad G_d by setting

$$G_d(k) = H_*(\mathcal{D}_{d+1}(k)) = H_*(C_k(\mathbb{R}^{d+1})).$$

The operad structure of G_d is induced from the operad structure of $\mathcal{D}_{d+1}$ in the obvious way.

Next, define the operad L_d by setting

$$L_d(k) = H_{(k-1)(d-1)}(\mathcal{D}_{d+1}(k)) = H_{(d-1)(n-1)}(C_k(\mathbb{R}^{d+1})).$$

These groups are the top non-zero homology groups of the configuration spaces. To define the structure maps apply the functor $H_{(n-1)(d-1)}$ to the structure maps

$$\mathcal{D}_d(k) \times \mathcal{D}_d(n_1) \times \cdots \times \mathcal{D}_d(n_k) \to \mathcal{D}_d(n), \quad n = n_1 + \cdots + n_k$$

in the topological operad $\mathcal{D}_d$. An exercise in dimension counting shows that this does indeed define maps

$$L_d(k) \otimes L_d(n_1) \otimes \cdots \otimes L_d(n_k) \to L_d(n)$$

and these are the structure maps in the operad L_d.

2.8. Graded Lie algebras. For reasons which are explained below we will now assume that the ground field K has characteristic different from 2 or 3. Recall the definition of a graded Lie algebra with Lie bracket of degree d. This is a graded vector space V equipped with a bilinear operation $[,] : V_i \otimes V_j \to V_{i+j+d}$ satisfying the following identities

(1) $[v, w] = -(-1)^{(|v|+d)(|w|+d)}[w, v]$
(2) $[u, [v, w]] = [[u, v], w] + (-1)^{(|u|+d)(|v|+d)}[v, [u, w]]$

In essence, a graded Lie bracket of degree d on V is a Lie bracket on V regraded by assigning degree $j + d$ to V_j. We follow up this point in §3.1. The first condition is then the *graded skew symmetry* of the bracket. The second is the *graded Jacobi identity*, it says that the operator $L_u = [u, -]$ is a graded derivation of degree $|u|+d$ of the bracket.

The problem with characteristic 2 is that skew symmetry does not necessarily tell us that $[x, x] = 0$ and the problem with characteristic 3 is that the Jacobi identity does not necessarily tell us that $[x, [x, x]] = 0$. To avoid the distraction of thinking about appropriate axioms for $[x, x]$ and $[x, [x, x]]$ in these special cases we exclude them.

It is clear that graded Lie algebras are algebras over an operad. It is one of the rather beautiful fundamental facts in the theory of operads that the operad which governs the structure of graded Lie algebras with bracket of degree d is the operad L_d constructed in the previous section using the top non-zero homology groups of configuration spaces.

THEOREM 2.4. *Suppose the ground field K has characteristic not equal to 2 or 3. A graded vector space V is an algebra over the operad L_d if and only if V is a graded Lie algebra with Lie bracket of degree d.*

This result is implicit in [**2**]. The idea behind the proof is this. We know that

$$L_d(2) = H_d(C_2(\mathbb{R}^{d+1}))$$

is one dimensional and the generator is the homology class dual to the cohomology class

$$\omega_{1,2} \in H^d(C_2(\mathbb{R}^{d+1})).$$

This gives us a bilinear operation, $\{,\}$ which, in view of the fact that $\omega_{2,1} = (-1)^d\omega_{1,2}$, satisfies

$$\{x, y\} = (-1)^{|x||y|+d}\{y, x\}.$$

The Lie bracket of degree d is given by

$$[x, y] = (-1)^{d|x|}\{x, y\}.$$

The Jacobi identity is ultimately a consequence of the relation

$$\omega_{1,2}\omega_{2,3} + \omega_{2,3}\omega_{3,1} + \omega_{3,1}\omega_{1,2} = 0$$

in $H^{2d}(C_3(\mathbb{R}^{d+1}))$.

2.9. Gerstenhaber algebras. A vector space V is a Gerstenhaber d-algebra if it is equipped with a Lie bracket $[,]$ of degree d satisfying the graded skew symmetry condition and the graded Jacobi identity, as in §2.8, and in addition it has a graded commutative product of degree zero such that

$$[u, vw] = [u, v]w + (-1)^{|v|(|u|+d)}b[a, c]$$

As in the case of Lie algebras of degree d we will assume that the ground field K has characteristic different from 2 or 3.

Thus in a Gerstenhaber d algebra we have a bracket of degree d, a commutative product of degree 0, and the operators $L_a = [a, -]$ are graded derivations of degree $|a|+d$ of both the bracket and the product. Once more it is clear that Gerstenhaber algebras are algebras over an operad. The next theorem identifies this operad explicitly.

THEOREM 2.5. *Suppose the ground field K has characteristic different from 2 or 3. A vector space V is a Gerstenhaber d-algebra if and only if V is an algebra over the operad G_d.*

For a proof see, for example [**7**], note that it is implicit in [**2**]. The idea is as follows. By definition

$$G_d(2) = H_*(C_2(\mathbb{R}^{d+1})).$$

The generator of the group $H_0(C_2(\mathbb{R}^{d+1}))$ defines the product, and the generator of the group $H_d(C_2(\mathbb{R}^{d+1}))$ defines the bracket. The relations between these operations are consequences of the structure of $H_*(C_3(\mathbb{R}^{d+1}))$.

2.10. Operads and structures up to homotopy. One fundamental idea in homological algebra is the use of resolutions: given a module M over a ring R a resolution of M is a chain complex P of R modules (that is a differential graded R module) equipped with a map $P \to M$ which induces an isomorphism in homology $H_*(P) \cong M$. In order for a resolution to be useful it is necessary to impose some kind of condition, like projective, on the modules in the chain complex P. We can use this idea in the theory of operads as follows.

DEFINITION 2.6. Let $\mathfrak{a}$ be a graded operad. A resolution of $\mathfrak{a}$ is a differential graded operad $\mathfrak{b}$ equipped with a map of operads $\mathfrak{b} \to \mathfrak{a}$ which induces an isomorphism in homology $H_*(\mathfrak{b}) \to \mathfrak{a}$.

Now we would like some kind of analogue of the projective condition appropriate for working with operads. The main difficulty is that operads do not form an abelian category so we cannot use the concepts of homological algebra, in particular projective objects, in the category of operads.

To get further, let us borrow uncritically some of the concepts and language of homotopy theory. A map of differential graded operads which induces an isomorphism in homology will be called a weak homotopy equivalence and we use the term weak homotopy equivalence for the equivalence relation between operads generated, in the usual way, by weak homotopy equivalences. Then a resolution $\mathfrak{b} \to \mathfrak{a}$ of a graded operad $\mathfrak{a}$ is a differential graded operad which is weakly homotopy equivalent to $\mathfrak{a}$. Suppose we have two resolutions $\mathfrak{b} \to \mathfrak{a}$ and $\mathfrak{b}' \to \mathfrak{a}$. Then it follows that $\mathfrak{b}$ and $\mathfrak{b}'$ are weakly homotopy equivalent and we would like to ensure that there are maps of operads $\mathfrak{b} \to \mathfrak{b}'$ and $\mathfrak{b}' \to \mathfrak{b}$ which induce inverse isomorphisms in homology. In the usual homotopy theory of (simply connected) topological spaces we know that if two CW complexes are weakly homotopy equivalent, then they are homotopy equivalent. So we are looking for the analogues of CW complexes in the category of operads.

The appropriate general setting for the concepts of homotopy theory is Quillen's theory of homotopical algebra [**17**]. We will not give the details but we will summarize one or two key points, as much as anything to establish the appropriate language. The category of operads does indeed have the structure of a closed model category in the sense of [**17**]. This means we can do homotopy theory in the category of operads and talk about weak homotopy equivalences (defined as above), fibrations, cofibrations, and homotopy equivalences. In a closed model category the analogue of CW complexes are cofibrant objects. In particular if two cofibrant objects are weakly equivalent, than they are homotopy equivalent.

Now, given a cofibrant resolution $\mathfrak{b} \to \mathfrak{a}$ of an operad $\mathfrak{a}$ it seems natural to think of algebras over $\mathfrak{b}$ as '$\mathfrak{a}$ algebras up to homotopy'. This notion has two characteristic properties

(1) If V is an algebra over $\mathfrak{a}$ then using the map $\mathfrak{b} \to \mathfrak{a}$ it follows that V is an algebra over $\mathfrak{b}$ and therefore is an '$\mathfrak{a}$ algebra up to homotopy'.
(2) If V is a differential graded $\mathfrak{b}$ algebra then $H_*(V)$ becomes an algebra over $H_*(\mathfrak{b}) = \mathfrak{a}$. Thus the homology of an '$\mathfrak{a}$ algebra up to homotopy' is an $\mathfrak{a}$ algebra.

We will not discuss homotopical algebra and the closed model structure on the category of operads in these notes. However, in §5.2 we give the definition of an almost free differential graded operad. This is sufficient for our purposes as all the

resolutions we construct satisfy this condition. Furthermore, it is the case that an almost free operad is cofibrant. One of the main purposes of these notes is to construct almost free resolutions of a class of operads which include the operads governing the structure of associative algebras, commutative algebras, Lie algebras and Gerstenhaber algebras, and to look in some detail at the associated notions of algebras up to homotopy.

3. Some background homological algebra

We have seen in §2.4 that given an algebra A we get an operad I_A. Indeed the function $A \mapsto I_A$ defines a bijection between algebras and operads $\mathfrak{a}$ with the property that $\mathfrak{a}(n) = 0$ if $n \neq 1$. One of the themes in the general theory of operads is to think of operads as generalizations of algebras and to generalize standard constructions with algebras to constructions with operads. In this section we review some these standard constructions, in particular the tensor algebra, the tensor coalgebra, the bar construction, and the cobar construction. The main objective is to express these notions in a way which makes their generalization to operads as straightforward as possible.

3.1. Signs and gradings. To keep track of signs we will rigorously adopt the sign conventions of graded algebra and take great care to ensure that any identifications which involve a change of grading are made explicit so that their effect on signs does not get forgotten. The basic sign convention of graded algebra is that if two objects x and y are interchanged then the sign $(-1)^{|x||y|}$ is introduced.

If V and W are graded vector spaces then $V \otimes W$ is given the usual tensor product grading, that is

$$|v \otimes w| = |v| + |w|$$

where v and w are homogeneous elements of V and W respectively. If $f : V \to W$ is a homomorphism then f is homogeneous and has degree p if

$$f : V_i \to W_{i+p}.$$

We write $\mathrm{Hom}^p(V, W)$ for the space of homogeneous homomorphisms of degree p. Suppose $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$ are homogeneous, then $f \otimes g : V_1 \otimes V_2 \to W_1 \otimes W_2$ is defined by

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$$

and

$$|f \otimes g| = |f| + |g|.$$

One consequence of this sign convention is the following formula. Suppose that f and g are isomorphisms, then

$$(f \otimes g)^{-1} = (-1)^{|f||g|} f^{-1} \otimes g^{-1}.$$

If V is a graded vector space and k is an integer, let $\Sigma^k V$ be the graded vector space defined by

$$(\Sigma^k V)_n = V_{n-k}.$$

There is an isomorphism

$$\sigma^k : V \to \Sigma^k V$$

of degree k. When using this isomorphism σ^k we must remember to introduce signs corresponding to the fact that σ^k has degree k. If $f : V \to V$ is a linear map then we get a linear map $\Sigma^k f : \Sigma^k V \to \Sigma^k V$ in the obvious way.

Here is an example of these sign conventions which is particularly relevant. Suppose we have a bilinear product

$$\phi : V \otimes V \to V$$

of degree zero on a graded vector space V. Then we can convert this to a bilinear product

$$\tilde{\phi} : \Sigma^k V \otimes \Sigma^k V \to \Sigma^k V$$

of degree k by requiring the following diagram to commute

$$\begin{array}{ccc} V \otimes V & \xrightarrow{\phi} & V \\ {\scriptstyle \sigma^k \otimes \sigma^k}\downarrow & & \downarrow{\scriptstyle \sigma^k} \\ \Sigma^k V \otimes \Sigma^k V & \xrightarrow[\tilde{\phi}]{} & \Sigma^k V \end{array}$$

Thus $\tilde{\phi}$ is characterized by the equation

$$\sigma^k \circ \phi = \tilde{\phi} \circ \sigma^k \otimes \sigma^k.$$

Let us compute $\tilde{\phi}$ on elements. By definition

$$\sigma^k \phi(x \otimes y) = \tilde{\phi}(\sigma^k \otimes \sigma^k)(x \otimes y) = (-1)^{k|x|} \tilde{\phi}(\sigma^k x \otimes \sigma^k y)$$

and so we have the equation

$$\tilde{\phi}(\sigma^k x \otimes \sigma^k y) = (-1)^{k|x|} \sigma^k \phi(x \otimes y).$$

Now suppose the degree zero product ϕ on V is associative that is

$$\phi \circ (\phi \otimes 1) = \phi \circ (1 \otimes \phi).$$

A routine computation shows that ϕ is associative if and only if

$$\tilde{\phi} \circ (\tilde{\phi} \otimes 1) = (-1)^k \tilde{\phi} \circ (1 \otimes \tilde{\phi}).$$

If we assume that the degree zero product ϕ on V is a Lie bracket then similar computations show that the degree k product $\tilde{\phi}$ on $\Sigma^k V$ is a Lie bracket of degree k as defined in §2.8.

3.2. The tensor algebra. Let V be a vector space over the ground field K. Then recall that the tensor algebra of V is defined as follows. Use the notation

$$T^p(V) = V^{\otimes p}$$

for the p-th tensor power of V. Then there are natural isomorphisms

$$T^p(V) \otimes T^q(V) \to T^{p+q}(V)$$

$$(v_1 \otimes \cdots \otimes v_p) \otimes (w_1 \otimes \cdots w_q) \mapsto v_1 \otimes \cdots v_p \otimes w_1 \otimes \cdots \otimes w_q.$$

Using these isomorphisms we get a product on the space

$$T(V) = \sum_{i=0}^{\infty} T^p(V)$$

and this product is associative and unital. The *tensor algebra* of V is the algebra $A(V)$ with underlying vector space $T(V)$ and the above product. We use the notation $A(V)$ (rather than $T(V)$) to emphasize that we are using the algebra structure. Note that $A(V)$ is an augmented algebra; the augmentation is given by the projection $A(V) \to T^0(V) = K$.

For unital K-algebras B and C we will use the notation

$$\mathrm{Hom}_{\mathrm{alg}}(B, C)$$

for the set of homomorphisms of unital K algebras. If in addition B and C are augmented we use the notation

$$\mathrm{Hom}_{\mathrm{alg}^+}(B, C).$$

for the homomorphisms which preserve the augmentation. The next lemma tells us that $A(V)$ is the free algebra generated by V.

LEMMA 3.1. *Let $i_1 : V \to A(V)$ be the inclusion of the summand $T^1(V)$.*

(1) *The linear map i_1 induces an isomorphism*

$$\mathrm{Hom}_{alg}(AV, B) \to \mathrm{Hom}_K(V, B)$$

for any unital K-algebra B.

(2) *If B is an augmented unital K algebra then i_1 induces an isomorphism*

$$\mathrm{Hom}_{alg^+}(TV, B) \to \mathrm{Hom}_K(V, IB)$$

where IB is the augmentation ideal of B (the kernel of the augmentation $B \to K$).

Next we consider derivations of $A(V)$. Recall that a derivation of an algebra A is a linear map $D : A \to A$ such that

$$D(ab) = (Da)b + a(Db).$$

We will write $\mathrm{Der}(A)$ for the space of derivations of the algebra A. It is straightforward to determine the derivations of the tensor algebra $A(V)$.

LEMMA 3.2. *The inclusion $i_1 : V \to A(V)$ induces an isomorphism*

$$\mathrm{Der}(A(V)) \to \mathrm{Hom}_K(V, A(V)).$$

If V is a graded vector space then the tensor algebra $A(V)$ becomes a graded algebra where the degree of a homogeneous tensor is defined by

$$|v_1 \otimes \cdots \otimes v_p| = |v_1| + \cdots + |v_p|.$$

Note that the 'length' p of the tensor $v_1 \otimes \cdots \otimes v_p$ does not affect its grading. The isomorphisms of Lemma 3.2 extend to the graded case without any changes.

If B is a graded algebra then a graded derivation of degree p is a linear map $D : B \to B$ of degree p such that

$$D(ab) = D(a)b + (-1)^{p|a|}aD(b).$$

The space of derivations of degree p is denoted by $\mathrm{Der}^p(B)$. Then if V is a graded vector space, the graded derivations of $A(V)$ are given by

$$\mathrm{Der}^p(A(V)) = \mathrm{Hom}^p(V, A(V)).$$

This is the natural extension of Lemma 3.2 to the graded case.

3.3. The tensor coalgebra. We now consider the coalgebra 'dual' to the tensor algebra. The duality between algebras and coalgebras is slightly awkward because of the fact that (linear) duals and tensor products do not commute: if V and W are vector spaces over K there is always a linear map $V^* \otimes W^* \to (V \otimes W)^*$ but this map is an isomorphism if and only if both V and W are finite dimensional. A consequence of this is that if C is a coalgebra then it is the case that its linear dual C^* is an algebra with product given by

$$C^* \otimes C^* \to (C \otimes C)^* \to C^*$$

where the second arrow is the linear map induced by the coproduct in C. However, if A is an algebra, then A^* with the coproduct induced by the product in A is a coalgebra if A is finite dimensional but not, in general, a coalgebra if A is infinite dimensional.

Nonetheless there is a coalgebra $C(V)$ which is 'pre-dual' to the tensor algebra $A(V)$ in the sense that there is a canonical homomorphism of algebras

$$A(V^*) \to (C(V))^*$$

which is an isomorphism if V is finite dimensional. The purpose of this section is to describe this coalgebra and one or two of its more useful properties.

Let V be a vector space. Suppose $n = p + q$ and let

$$\psi_{p,q} : T^n(V) \to T^p(V) \otimes T^q V$$

be the canonical isomorphism: explicitly

$$\psi_{p,q}(v_1 \otimes \cdots \otimes v_n) = (v_1 \otimes \cdots \otimes v_p) \otimes (v_{p+1} \otimes \cdots \otimes v_n).$$

The terms $\psi_{0,n}$ and $\psi_{n,0}$ are defined by

$$\psi_{0,n}(v_1 \otimes \cdots \otimes v_n) = 1 \otimes (v_1 \otimes \cdots \otimes v_n)$$

$$\psi_{n,0}(v_1 \otimes \cdots \otimes v_n) = (v_1 \otimes \cdots \otimes v_n) \otimes 1$$

where 1 is the unit in the scalars $K = T^0(V)$. These linear maps define a coproduct

$$\psi : T(V) \to T(V) \otimes T(V),$$

$$\psi(v_1 \otimes \cdots \otimes v_p) = \sum_{p+q=n} \psi_{p,q}(v_1 \otimes \cdots \otimes v_n).$$

This is the *Alexander-Whitney coproduct.* It is straightforward to check that this coproduct is coassociative, that is

$$\psi \circ (\psi \otimes 1) = \psi \circ (1 \otimes \psi).$$

The *tensor coalgebra* of V is the coalgebra $C(V)$ with underlying vector space $T(V)$ and with coproduct given by the Alexander-Whitney coproduct. As in the case of the tensor algebra we use the notation $C(V)$ to emphasize that we are considering the coalgebra structure.

The natural linear maps

$$T^p(V^*) \to T^p(V)^*$$

induce a linear map

$$A(V^*) \to C(V)^*$$

and it is easy to see that this is a homomorphism of algebras which is an isomorphism if V is finite dimensional.

If C is a coalgebra with coproduct $\psi : C \to C \otimes C$, a *counit* for C is a linear map $\eta : C \to K$ such that the composites

$$C \xrightarrow{\psi} C \otimes C \xrightarrow{\eta \otimes 1} K \otimes C = C,$$

$$C \xrightarrow{\psi} C \otimes C \xrightarrow{1 \otimes \eta} C \otimes K = C$$

are both the identity. A *coaugmentation* for a counital coalgebra C is a map of coalgebras $\epsilon : K \to C$ such that $\eta\epsilon = 1$. Here K has the coalgebra structure defined by the canonical isomorphism $K \to K \otimes K$. The tensor coalgebra $C(V)$ is counital and coaugmented: the counit $C(V) \to K$ is the projection onto $T^0(V) = K$ and the coaugmentation $K \to C(V)$ is the inclusion of the summand $T^0(V) = K$.

We now look for the pre-dual of the isomorphism in Lemma 3.1 of the previous section. This should determine all the maps of counital coalgebras

$$f : C \to C(V).$$

Clearly such a map is uniquely determined by the composites

$$f_k = p_k \circ f$$

where, $p_k : C(V) \to T^k(V)$ is the projection onto the summand $T^k(V)$. Since we are assuming f preserves counits it follows that

$$f_0 : C \to K$$

is just the counit. Using the fact that f is a homomorphism of coalgebras we now derive a formula for f_k in terms of f_1. First define

$$\psi^{(k)} : C \to T^k(C), \quad k \geq 2$$

inductively as follows: $\psi^{(2)} = \psi$ and $\psi^{(k)}$ is the composite

$$C \xrightarrow{\psi} C \otimes C \xrightarrow{1 \otimes \psi^{(k)}} C \otimes T^k(C) = T^{k+1}(C).$$

This map $\psi^{(k)}$ is the pre-dual of the iterated multiplication map $T^k(A) \to A$ in an algebra A in the sense that if $A = C^*$ is the dual algebra of C then $(\psi^{(k)})^*$ is the multiplication map of A iterated k times. Since f is a map of coalgebras it follows that

$$f_k = T^k(f_1) \circ \psi^{(k)}.$$

Thus the map f is uniquely determined by the single map $f_1 : C \to V$.

Suppose now we are given a map $\phi : C \to V$. Is there a homomorphism of counital coalgebras $f : C \to C(V)$ such that $f_1 = \phi$? If there is then it must follow that f_0 is the counit, $f_k = T^k(\phi) \circ \psi^{(k)}$, and

$$f = \sum f_k.$$

However this map f does not necessarily map to the direct sum of the $T^k(V)$ since we cannot assume that $f_k(x)$ is zero for all but a finite number of values of k.

This will be true however if the coalgebra C is conilpotent. The notion of a *conilpotent coalgebra* is the pre-dual of the notion of an augmented algebra whose augmentation ideal is nilpotent. Here is the precise definition of a conilpotent coalgebra C. We must assume that the coalgebra C is counital and coaugmented.

Now define $\bar{C}$ to be the cokernel of ϵ and define $\bar{\psi}^{(k)} : \bar{C} \to T^k(\bar{C})$ to be the unique map such that the following diagram commutes.

$$\begin{array}{ccc} C & \xrightarrow{\psi^{(k)}} & T^k(C) \\ \downarrow & & \downarrow \\ \bar{C} & \xrightarrow[\bar{\psi}^{(k)}]{} & T^k(\bar{C}) \end{array}$$

This map $\bar{\psi}^{(k)}$ is the pre-dual of the restriction of the iterated multiplication map $A^{\otimes k} \to A$ in an augmented algebra A to the k-fold tensor product of the augmentation ideal.

An element x in the coalgebra C is a *conilpotent element* if there is an integer k such that

$$\bar{\psi}^{(k)}(x) = 0.$$

This is the analogue for coalgebras of a nilpotent element in the augmentation ideal of an augmented algebra. A coalgebra C is conilpotent if every element of C is conilpotent. This definition of conilpotence is sufficient for our purposes however there is a stronger notion: we could ask that there is an integer k such that $\bar{\psi}^{(k)}(x) = 0$ for all $x \in C$. The dual of this stronger notion of conilpotence is that the augmentation ideal in the dual algebra is a nilpotent ideal.

The tensor coalgebra $C(V)$ is conilpotent: if $x \in T^p(V) \subset C(V)$ then

$$\bar{\psi}^{(p)}(x) = 0.$$

We now have the following theorem.

THEOREM 3.3. *Let C be conilpotent counital coaugmented coalgebra. Then the projection $p_1 : C(V) \to V$ onto the summand $V = T^1(V)$ induces an isomorphism*

$$\mathrm{Hom}_{coalg^+}(C, TV) \to \mathrm{Hom}_K(\bar{C}, V).$$

Note that in the statement of this theorem $\mathrm{Hom}_{\mathrm{coalg}^+}$ means the set of coalgebra homomorphisms which preserve the counit and the coaugmentation.

There is the corresponding version of this result if V is a graded vector space and $C(V)$ is graded using the tensor product grading, i.e.

$$|c_1 \otimes \cdots \otimes c_p| = |c_1| + \cdots + |c_p|.$$

Note once more that the length of the tensor plays no role in the definition of the grading. Theorem 3.3 extends to this case in the obvious way.

As in the ungraded case the coalgebra $C(V)$ is conilpotent. Here is another useful class of conilpotent graded coalgebras. Suppose C is a graded coaugmented counital coalgebra which is bounded below, that is $C_j = 0$ for all j with $j \leq N$ for some $N \in \mathbb{Z}$, then C is conilpotent. In this case if $C_j = 0$ for $j \leq N$ and $x \in C_p$ then $\bar{\psi}^{(N+p)}(x) = 0$.

For example, we can apply Theorem 3.3 to study homomorphisms of counital coaugmented coalgebras

$$f : C(V) \to C(W).$$

Such a homomorphism is uniquely determined by its components

$$f_k^n = p_k f i_n$$

where $p_k : C(W) \to T^k(W)$ is the projection and $i_n : T^n(V) \to C(V)$ is the inclusion. For each integer $n \geq 0$ let $\phi_n : T^n(V) \to W$ be the linear map

$$\phi_n = f_1^n = p_1 f i_n.$$

Note that $\phi_0 = 0$ since f preserves the counit and the coaugmentation.

LEMMA 3.4.

$$f_k^n = \sum_{n_1+\cdots+n_k=n} \phi_{n_1} \otimes \cdots \otimes \phi_{n_k}$$

To prove this, let $f_k = p_k f$. Then, as above, since f is a map of coalgebras

$$f_k = T^k(f_1)\psi^{(k)}.$$

However

$$\psi^{(k+1)} = (1 \otimes \psi^{(k)})\psi$$

and so

$$f_{k+1} = (\phi \otimes f_k)\psi$$

and a straightforward inductive argument completes the proof. This lemma will prove useful in §6.

Next we consider coderivations of the coalgebra $C(V)$. Recall that a coderivation of coalgebra C is a linear map $L : C \to C$ such that the following diagram commutes

$$\begin{array}{ccc} C & \xrightarrow{L} & C \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ C \otimes C & \xrightarrow[L\otimes 1+1\otimes L]{} & C \otimes C. \end{array}$$

The next lemma determines the space of coderivations of the coalgebra $C(V)$, compare [6].

LEMMA 3.5. *The linear map* $p_1 : C(V) \to V$ *induces an isomorphism*

$$\mathrm{Coder}(C(V)) \to \mathrm{Hom}(C(V), V)$$

As in the previous section this result extends to the case where V is a graded vector space to tell us that

$$\mathrm{Coder}^p(C(V)) = \mathrm{Hom}^p(C(V), V)$$

where Coder^p is the space of coderivations of degree p.

Given a linear map $f : T^k(V) \to V$ then it will be useful to have a formula for the coderivation

$$L_f : C(V) \to C(V)$$

defined by the linear map

$$C(V) \xrightarrow{p_k} T^k(V) \xrightarrow{f} V$$

This coderivation is zero on the summand $T^j(V)$ if $j < k$ and if $j \geq k$ then on the summand $T^j(V)$ it is given by

$$L_f = \sum_{i=0}^{j-k} 1^{\otimes i} \otimes f \otimes 1^{\otimes(j-i-k)}.$$

3.4. The bar construction and the cobar construction.

3.4.1. *The bar construction.* Now suppose A is a differential graded algebra. The bar construction on A is the differential graded coalgebra defined as follows. First form the graded vector space ΣA, and now form the tensor coalgebra $C(\Sigma A)$. Notice that in $C(\Sigma A)$ the length of the tensor has been used in the grading:

$$|\sigma a_1 \otimes \cdots \otimes \sigma a_p| = |a_1| + \cdots + |a_p| + p.$$

Now consider the linear maps

$$C(\Sigma A) \xrightarrow{p_1} \Sigma A \xrightarrow{\Sigma d} \Sigma A$$

$$C(\Sigma A) \xrightarrow{p_2} \Sigma A \otimes \Sigma A \xrightarrow{\tilde{m}} \Sigma A$$

Here $p_k : C(\Sigma A) \to (\Sigma A)^{\otimes k} = T^k(\Sigma A)$ is the projection, d is the differential in A, $m : A \otimes A \to A$ is the product in A, and $\tilde{m}$ is the degree 1 bilinear product on ΣA defined using the constructions of §3.1. Both these linear maps have degree -1. By Lemma 3.5 they define coderivations

$$L_d, L_m : C(\Sigma A) \to C(\Sigma A)$$

of degree -1.

LEMMA 3.6.

$$(L_d)^2 = 0, \quad (L_m)^2 = 0, \quad L_d L_m + L_m L_d = 0$$

It follows that

$$\partial = L_d + L_m$$

is a coderivation of $C(\Sigma A)$ of degree -1 with $\partial^2 = 0$. Therefore $C(\Sigma A)$ equipped with differential ∂ is a differential graded coalgebra. This differential graded coalgebra is the *bar construction* on A and is denoted by $B(A)$.

We can convert this implicit definition into explicit formulas using the formula for the coderivations L_f given in the previous section. It is a good exercise to work this through in detail, including the signs, to get the usual formulas for the bar construction on a differential graded algebra.

Next we ask how to construct a map of differential graded coalgebras $f : C \to B(A)$. As in the previous section we assume that C is counital, coaugmented, conilpotent, and the map f preserves counits and coaugmentations. Now the underlying graded coalgebra of $B(A)$ is the tensor coalgebra $C(\Sigma A)$. In view of Lemma 3.3 this map is uniquely determined by the linear map

$$f_1 : C \to \Sigma A$$

defined to be the composite

$$C \xrightarrow{f} B(A) \xrightarrow{p_1} \Sigma A.$$

The condition that f preserves the counit and the coaugmentation gets translated into the condition that f_1 vanishes on the image of the coaugmentation.

Now we formulate the condition that f commutes with the differentials. Since C is a coalgebra and A is an algebra the space $\mathrm{Hom}(C, A)$ has a product: given $f, g \in \mathrm{Hom}(C, A)$ their product, *the cup product* $f \cup g$, is composite

$$C \xrightarrow{\psi} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A$$

where ψ is the coproduct in C and m is the product in A. Since C and A are differential graded objects $\mathrm{Hom}(C, A)$ inherits a differential. The cup product and this natural differential make $\mathrm{Hom}(C, A)$ into a differential graded algebra.

Now let $\phi : C \to A$ be a linear map which vanishes on the image of the coaugmentation. We are assuming that C is counital coagumented and conilpotent so there is a unique map $f : C \to B(A)$ of graded counital coalgebras such that $f_1 = \phi$. Then this map is a map of differential graded coalgebras (i.e. it commutes with the differentials) if and only if $\phi \in \mathrm{Hom}(C, A)$ satisfies the equation

$$d\phi + \phi \cup \phi = 0$$

where d is the differential in $\mathrm{Hom}(C, A)$. We formulate this precisely as the following theorem.

THEOREM 3.7. *Let A be a unital differential graded algebra and let C be a conilpotent counital coaugmented differential graded coalgebra. The projection $p_1 : B(A) \to \Sigma A$ defines a bijection from $\mathrm{Hom}_{dgcoalg^+}(C, B(A))$ to the set of $\phi \in \mathrm{Hom}^{-1}(C, A)$ such that*

$$d\phi + \phi \cup \phi = 0.$$

3.4.2. *The cobar construction.* Now suppose C is a differential graded coalgebra. The cobar construction $F(C)$ is the differential graded algebra which is the pre-dual of the bar construction in the following sense. There is linear map

$$B(C^*) \to F(C)^*$$

which is an isomorphism of differential graded coalgebras if C is finite dimensional.

The detailed construction of $F(C)$ goes as follows. First form the graded vector space $\Sigma^{-1}C$, and next form the tensor algebra $A(\Sigma^{-1}C)$. Now consider the linear maps

$$\Sigma^{-1}C \xrightarrow{\Sigma^{-1}d} \Sigma^{-1}C \xrightarrow{i_1} A(\Sigma^{-1}C)$$

$$\Sigma^{-1}(C) \xrightarrow{\tilde{\psi}} T^2(\Sigma^{-1}C) \xrightarrow{i_2} A(\Sigma^{-1}C)$$

Here $i_k : T^k(\Sigma^{-1}C) \to A(\Sigma^{-1}C))$ is the inclusion of the summand $T^k(\Sigma^{-1}C)$; d is the differential in C; and $\tilde{\psi}$ is the coproduct of degree -1 defined by the coproduct $\psi : C \to C \otimes C$. These two linear maps both have degree -1 and so, using Lemma 3.2, they define derivations

$$D_d, D_\psi : A(\Sigma^{-1}C) \to A(\Sigma^{-1}C).$$

LEMMA 3.8.

$$D_d^2 = 0, \quad D_\psi^2 = 0, \quad D_d D_\psi + D_\psi D_d = 0$$

It follows that

$$\partial = D_d + D_\psi$$

is a derivation of $A(\Sigma^{-1}C)$ with $\partial^2 = 0$. Therefore $A(\Sigma^{-1}C))$ equipped with differential ∂ is a differential graded algebra. This differential graded algebra is the *cobar construction* on C and is denoted by $F(C)$.

Let A be a differential graded algebra and let us now consider the space of homomorphisms of differential graded algebras $F(C) \to A$. Such a homomorphism is uniquely determined by its restriction to C and just as in the case of the bar construction we see that there is a one-one correspondence between homomorphisms of differential graded algebras $F(C) \to A$ and the space of homomorphisms $\phi : C \to A$ which satisfy the equation

$$d\phi + \phi \cup \phi = 0.$$

More precisely we have the following theorem.

THEOREM 3.9. *Let A be an augmented unital differential graded algebra and let C be a counital coaugmented differential graded coalgebra. Then the inclusion of $\Sigma^{-1}C$ in $F(C)$ induces a bijection between the set $\mathrm{Hom}_{dgalg^+}(F(C), A)$ to the set of $\phi \in \mathrm{Hom}^{-1}(C, A)$ such that*

$$d\phi + \phi \cup \phi = 0.$$

3.5. The adjointess of the bar construction and the cobar construction. Now suppose that A is an augmented unital differential graded algebra and C is a conilpotent coaugmented counital differential graded coalgebra. Combining Theorem 3.7 and Theorem 3.9 shows that there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{dgalg}^+}(F(C), A) = \mathrm{Hom}_{\mathrm{dgcoalg}^+}(C, B(A))$$

provided C is conilpotent. In particular since $B(A)$ is conilpotent, this gives a natural map of differential graded algebras

$$F(B(A)) \to A,$$

corresponding to the identity in $\mathrm{Hom}(B(A), B(A))$. If C is conilpotent we get a natural map of differential graded coalgebras

$$C \to F(B(C))$$

corresponding to the identity map in $\mathrm{Hom}(F(C), F(C))$. Under suitable connectivity conditions on A and C these maps are quasi-isomorphisms, that is they induce isomorphims in homology.

In particular, if the differential graded algebra A is zero in negative degrees it follows that

$$F(B(A)) \to A$$

is a quasi-isomorphism. Now notice that the algebra $F(B(A))$ is almost free in the sense that if we forget the differential on $F(B(A))$ it is a free algebra. Therefore $F(B(A)) \to A$ is an almost free resolution of A since $F(B(A))$ is an almost free differential graded algebra and the map $F(B(A)) \to A$ is an isomorphism in homology. We would like to be able to make the corresponding constructions for operads to get almost free resolutions.

4. Operads and deformation quantisation

4.1. The main problem of deformation quantization. We begin by recalling the fundamental problem in deformation quantization. Let M be a smooth manifold. Then M is a *Poisson manifold* if the ring $C^\infty(M)$ of smooth complex valued functions on M is equipped with a bilinear operation

$$\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

satisfying the following properties.

(1) The bracket $\{,\}$ is bilinear over $\mathbb{C}$.
(2) The bracket $\{,\}$ is skew symmetric,

$$\{f, g\} = -\{g, f\}$$

(3) The bracket $\{,\}$ satisfies the Jacobi identity

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}.$$

(4) The bracket $\{,\}$ is a derivation of the product

$$\{f, gh\} = g\{f, h\} + \{f, g\}h.$$

In the language of §2.9, M is a Poisson manifold if $C^\infty(M)$ is a G_0 algebra. In the literature G_0-algebras are usually called Poisson algebras because of this example.

Fix f and consider the operator $L_f : C^\infty(M) \to C^\infty(M)$ defined by

$$L_f(g) = \{f, g\}.$$

Then, since the bracket is a derivation of the product, it follows that L_f is a differential operator. Using the symmetry of the bracket it follows that if we fix g the operator R_g defined by

$$R_g(f) = \{f, g\}$$

is a differential operator. Thus the bracket $\{f, g\}$ is a bidifferential operator in f and g. This ensures that there is a skew symmetric bi-vector field $P \in C^\infty(\Lambda^2 TM)$ is such that

$$\{f, g\} = \langle P, df \wedge dg \rangle.$$

The notation is as follows: TM is the tangent bundle of M; $\Lambda^2 TM$ is the second exterior power of TM; for any smooth vector bundle E over M, $C^\infty(E)$ is the space of smooth sections of E; df and dg are the one forms given by the derivatives of f and g respectively; $\wedge$ is the exterior product of forms; and $\langle , \rangle$ is the natural pairing $C^\infty(\Lambda^2 M) \times C^\infty(\Lambda^2 T^* M) \to C^\infty(M)$. The bi-vector field P is usually known as the *Poisson tensor* of M.

We will denote the ring of formal power series in an indeterminate t with coefficients in ring R by $R[[t]]$. If M is a Poisson manifold, then a *$*$-product* for M is a product operation

$$C^\infty(M)[[t]] \times C^\infty(M)[[t]] \to C^\infty(M)[[t]], \quad (a, b) \mapsto a * b$$

which is bilinear over $\mathbb{C}[[t]]$ and has the following properties.

(1) The product $*$ is associative.

(2) If $f, g \in A \subset A[[t]]$ then

$$f * g = fg \mod t, \quad f * g - g * f = t\{f, g\} \mod t^2$$

It is usual to assume that the $*$-product is given by bi-differential operators in the following sense. Suppose $f, g \in C^\infty(M)$ and write out the power series $f * g$ in the form

$$f * g = fg + tC_1(f, g) + t^2 C_2(f, g) + \cdots.$$

Then we will say that the $*$ product is a *bi-differential $*$-product* if each of the operators $C_j(f, g)$ is a bidifferential operator.

The main problem of deformation quantization is this. Let M be a Poisson manifold. Does there exist a bi-differential $*$ product on $C^\infty(M)[[t]]$?

4.2. The Gerstenhaber bracket. Let V be a vector space. The Gerstenhaber bracket is a bilinear operation

$$[,] = [,]_G : \mathrm{Hom}(T^k(V), V) \otimes \mathrm{Hom}(T^l(V), V) \to \mathrm{Hom}(T^{k+l-1}(V), V).$$

The formula for this bracket is given in many places, see for example [**5**] and [**7**]. Before we consider the definition of the Gerstenhaber bracket we summarize its main properties.

(1) If $f \in \mathrm{Hom}(T^k(V), V)$, $g \in \mathrm{Hom}(T^l(V), V)$ then

$$[f, g] = -(-1)^{(k-1)(l-1)}[g, f]$$

(2) If $f \in \mathrm{Hom}(T^k(V), V)$, $g \in \mathrm{Hom}(T^l(V), V)$, $h \in \mathrm{Hom}(T^m(V), V)$ then

$$[f, [g, h]] = [[f, g], h] + (-1)^{(k-1)(l-1)}[g, [f, h]].$$

The following lemma shows that the Gerstenhaber bracket has a role to play in deformation quantization.

LEMMA 4.1. *Let $\phi : V \otimes V \to V$ be a bilinear operation. Then ϕ is associative if and only if $[\phi, \phi] = 0$.*

Now we turn to the definition of the Gerstenhaber bracket. Given two maps $f \in \mathrm{Hom}(T^k(V), V)$ and $g \in \mathrm{Hom}(T^l(V), V)$ define

$$f \circ g \in \mathrm{Hom}(T^{k+l-1}(V), V)$$

by

$$f \circ g = \sum_{i=0}^{k+l-1} \epsilon_{i,k,l} f(1^{\otimes i} \otimes g \otimes 1^{k+l-i-1})$$

where $\epsilon_{i,k,l}$ is a sign. Now define the bracket by

$$[f, g] = f \circ g - (-1)^{(k-1)(l-1)} g \circ f$$

The key is to get the signs right, that is define the signs $\epsilon_{i,k,l}$ so that the bracket does indeed have the above properties. We will do this implicitly, using an idea originally due to Stasheff.

The first thing to notice is that we really want to say that if $f \in \mathrm{Hom}(T^k(V), V)$ then f has degree $k-1$. We can achieve this by the following device. Regard V as a graded vector space which is zero in degrees other than 0. Now form the graded vector space $\Sigma^{-1}V$, so $\Sigma^{-1}V$ is V in degree -1 and zero in all other degrees. Then

$$\mathrm{Hom}^{k-1}(T(\Sigma^{-1}V), \Sigma^{-1}V) \cong \mathrm{Hom}(T^k(V), V)$$

and so $\mathrm{Hom}(T^k(V), V)$ is the degree $k-1$ part of $\mathrm{Hom}(T(\Sigma^{-1}V), \Sigma^{-1}V)$. The next thing to notice is that the expression for $f \circ g$ is reminiscent of the formula in §3.3 for coderivations of the tensor coalgebra. From Lemma 3.5, it follows that

$$\mathrm{Coder}^i(C(\Sigma^{-1}V)) = \mathrm{Hom}^i(C(\Sigma^{-1}V), \Sigma^{-1}V)$$

and since

$$\mathrm{Hom}^i(C(\Sigma^{-1}V, \Sigma^{-1}V) = \mathrm{Hom}(T^{i-1}(V), V)$$

we see that

$$\mathrm{Coder}^i(C(\Sigma^{-1}V)) = \mathrm{Hom}(T^{i-1}(V), V)$$

The graded commutator of graded coderivations of the tensor coalgebra $C(\Sigma^{-1}V)$ defines a bilinear operation

$$\mathrm{Hom}(T^k(V), V) \otimes \mathrm{Hom}(T^l(V), V) \to \mathrm{Hom}(T^{k+l-1}(V), V).$$

This bilinear operation is the Gerstenhaber bracket. If we work out an explicit formula for this bilinear operation using the sign conventions of graded algebra we get the standard formula for the Gerstenhaber bracket.

Notice that this construction of the Gerstenhaber bracket extends in a quite straightforward way to the case where V is a graded vector space.

4.3. Hochschild cohomology. Let A be an algebra. As in the previous section regard A as a graded object which is zero in degrees other than zero. Now form the graded vector space $\Sigma^{-1}A$, so $\Sigma^{-1}A$ is A in degree -1 and zero in all other degrees. Now consider

$$C(A,A) = \mathrm{Hom}(C(\Sigma^{-1}A, A).$$

Note that

$$C^k(A,A) = \mathrm{Hom}^k(C(\Sigma^{-1}A), A) = \mathrm{Hom}(T^k(A), A).$$

The graded vector space $C(A,A)$ has the following structure.

(1) It is a graded algebra. The product is the usual product in the space of homomorphisms from a coalgebra to an algebra: given $f, g : C(\Sigma^{-1}A) \to A$ their product, the cup product $f \cup g$,is defined to be the composite

$$C(\Sigma^{-1}A) \xrightarrow{\psi} C(\Sigma^{-1}A) \otimes C(\Sigma^{-1}A) \xrightarrow{f\otimes g} A \otimes A \xrightarrow{m} A$$

where ψ is the coproduct in $C(\Sigma^{-1}A)$ and m is the product in A.

(2) There is a graded Lie bracket of degree -1

$$[,] : C^p(A,A) \otimes C^q(A,A) \to C^{p+q-1}(A,A).$$

This is the Gerstenhaber bracket of the previous section. This bracket makes makes $C(A,A)$ into a graded Lie algebra with Lie bracket of degree -1.

Unfortunately, a couple of computations show that there is no good formula for

$$[f, g \cup h]$$

and it is not the case that $C(A,A)$ is a Gerstenhaber (-1) algebra. However, after we take cohomology with respect to a suitable codifferential we do get a Gerstenhaber (-1) algebra. The codifferential is the Hochschild coboundary operator b.

The simplest way to describe the Hochschild coboundary operator is as follows. Let $m : A \otimes A \to A$ be the multiplication map. Now let

$$m \in C^2(A,A)$$

be the element defined by the product $m : A \otimes A \to A$. Now define

$$b : C(A,A) \to C(A,A), \quad b(f) = [m, f].$$

Since $m \in C^2(A,A)$ and the bracket has degree -1 this operator b has degree 1. Now using the graded Jacobi identity for the Gerstenhaber bracket,

$$[m,[m,f]] = [[m,m],f] - [m,[m,f]],$$

and so

$$2[m,[m,f]] = [[m,m]],f].$$

Using Lemma 4.1 the associativity of m tells us that

$$[m,m] = 0$$

so provided the characteristic of the ground field K is not 2

$$b^2 = 0.$$

This is the *Hochschild coboundary operator* and it is a good exercise to derive the usual formula for b from this description. It is straightforward to check that b is a derivation of degree -1 of the cup-product and the Gerstenhaber bracket, that is

$$b(f \cup g) = (bf) \cup g + (-1)^{|f|} f \cup bg$$

$$b[f,g] = [bf,g] + (-1)^{|f|}[f,bg]$$

Now we form the *Hochschild cohomology* of A

$$H^*(A,A) = H^*(C(A,A),b)$$

The product and the bracket descend to cohomology.

THEOREM 4.2. *The cup-product and the Gerstenhaber bracket define the structure of a Gerstenhaber* (-1) *algebra on* $H^*(A,A)$.

The Gerstenhaber bracket is a Lie bracket of degree -1 at the cochain level so the fact that the Gerstenhaber bracket is a Lie bracket of degree -1 in cohomology follows immediately. It is not too difficult to prove that the cup-product is graded commutative in cohomology. The hard part of the theorem is proving that the Gerstenhaber bracket is a graded derivation of the cup-product, see [**5**] for a proof.

4.4. Formal deformations and Hochschild cohomology. We now assume the characteristic of the ground field K is not 2.

DEFINITION 4.3. Let A be an associative algebra over a field K. A *formal deformation* of the product on the algebra A is a $K[[t]]$ bilinear product $* : A[[t]] \times A[[t]] \to A[[t]]$ which is associative and satisfies the following condition

$$a * b = ab \mod t, \quad \text{if } a,b \in A \subset A[[t]].$$

If $*$ is formal deformation of the product on the associative algebra A it is uniquely determined by the linear map

$$\theta : A \otimes A \to A[[t]]$$

defined by

$$\theta(a \otimes b) = a * b.$$

Since $a * b = ab \mod t$ for $a,b \in A \subset A[[t]]$ it follows that

$$\theta(a,b) = m(a,b) + C(a,b)$$

where $m : A \otimes A \to A$ is the product map in A and

$$C \in C^2(A,A)[[t]], \quad C = 0 \mod t.$$

Here, $C^2(A,A)$ is the degree 2 term in the Hochschild cochain complex $C(A,A)$. Now in view of Lemma 4.1, the condition that the $*$ product is associative is that

$$[\theta,\theta] = 0 \in C^3(A,A)[[t]]$$

where $[,]$ is the $K[[t]]$ bilinear extension of the Gerstenhaber bracket. Expanding this equation using the fact that $\theta = m + C$ gives

$$[m+C, m+C] = [m,m] + [C,m] + [m,C] + [C,C] = 0.$$

Now use the following facts: $[m,m] = 0$ since m is associative; $[C,m] = [m,C]$ since the Gerstenhaber bracket is a Lie bracket of degree -1 and both m and C have degree 2. This shows that the $*$-product is associative if and only if

$$[m,C] + \frac{1}{2}[C,C] = 0.$$

Finally we use the fact that $[m,-] = b$ where b is the coboundary operator in $C(A,A)$ defined by the product m on A to translate the fact that the star product is associative into the following structural equation

$$bC + \frac{1}{2}[C,C] = 0 \quad \text{in } C^2(A,A)[[t]].$$

where here b is the $K[[t]]$ linear extension of the coboundary operator in $C(A,A)$. More precisely the above argument establishes the following theorem.

THEOREM 4.4. *Let A be an associative algebra over the ground field K. There is a one-one correspondence between formal deformations of the product of A and solutions of the equation*

$$bC + \frac{1}{2}[C,C] = 0$$

with $C \in C^2(A,A)[[t]]$.

Note that this structural equation makes sense in any differential graded Lie algebra. Recall that L is a differential graded Lie algebra if L has a Lie bracket, $[,]$ of degree p and a differential d of degree -1 such that

$$d[a,b] = (-1)^p \left([da,b] + (-1)^{|a|}[a,db] \right).$$

Then we can look for solutions to the equation

$$\beta C + \frac{1}{2}[C,C] = 0$$

in L.

4.5. Deformation quantization. Now let us apply this piece of general theory to the example $A = C^\infty(M)$ where M is a Poisson manifold with the following modifications. First, to take account of the condition that the $*$ product on a Poisson manifold is given by bidifferential operators we replace $\mathrm{Hom}(T^p(A),A)$ by

$$\mathrm{Hom}_{\mathrm{diff}}(C^\infty(M^p), C^\infty(M)$$

by which we mean those linear maps $\phi : C^\infty(M^p) \to C^\infty(M)$ which have the property that if we hold all but one of the variables constant the resulting operator is a differential operator; in other words those linear maps ϕ which are multi-differential operators. The cohomology of the resulting cochain complex will be denoted by $H^*_{\mathrm{diff}}(C^\infty(M), C^\infty(M))$. This differential Hochschild cohomology has been computed, this is the Hochschild-Kostant-Rosenberg theorem, see [**9**]. In fact

$$H^*_{\mathrm{diff}}(C^\infty(M), C^\infty(M)) \cong C^\infty(\Lambda TM)$$

where ΛTM is the exterior algebra bundle of the tangent bundle of M. This space is usually known as the space of poly-vector fields on M. Under this isomorphism the product in Hochschild cohomology corresponds to the usual exterior product of poly-vector fields. The Gerstenhaber bracket corresponds to the classical *Schouten bracket*. Recall that the Schouten bracket is the unique bracket on poly-vector fields which agrees with the Lie bracket on vector fields and and is a graded derivation of degree -1 of the exterior product.

Now suppose M is a Poisson manifold. Then we can regard the Poisson bracket $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ as a Hochschild 2-cochain. An easy computation shows it is a cocycle and under the above isomorphism of Hochschild cohomology with poly-vector fields it corresponds to the Poisson tensor. We can summarize

these identifications as follows:

$$H^*_{\text{diff}}(C^\infty(M), C^\infty(M)) \cong C^\infty(\Lambda TM)$$

$$\text{cup-product} \leftrightarrow \text{exterior product}$$

$$[,]_G \leftrightarrow [,]_S$$

$$\{,\} \leftrightarrow P$$

where $[,]_G$ is the Gerstenhaber bracket, $[,]_S$ is the Schouten bracket, $\{,\}$ is the Poisson bracket, and P is the Poisson tensor. Now it is not difficult to check that

$$[P, P]_S = 0.$$

For the rest of this section we set $A = C^\infty(M)$. Then $C^*(A, A)$ will mean differential Hochschild cochains and $H^*(A, A)$ will mean differential Hochschild cohomology. It follows that if we write $C \in H^2(A, A)$ for the Hochschild cohomology class corresponding to Poisson tensor P under the above isomorphism then

$$[C, C]_G = 0$$

This result tells us that C is a solution to the structural equation

$$d\phi + \frac{1}{2}[\phi, \phi] = 0$$

at the level of cohomology in the following sense. Regard $H^*(A, A)$ as a differential graded Lie algebra with differential zero and Lie bracket the Gerstenhaber bracket. Then the cohomology class C is a solution to the structural equation in $H^*(A, A)$.

Now suppose there is a map

$$H^*(A, A) \to C^*(A; A)$$

which is a map of differential graded Lie algebras which induces an isomorphism in cohomology. Then applying this map to the cohomology class C defined by the Poisson bracket gives a solution to the structure equation in the differential graded Lie algebra $C^*(A, A)$.

There is indeed a natural map

$$H^*(A, A) \to C^*(A, A)$$

of differential graded algebras which induces an isomorphism in cohomology. Here $H^*(A, A)$ is given the zero differential and the cup-product defines the product. This is part of the proof of the Hochschild-Kostant-Rosenberg theorem. However, this map does not preserve the bracket.

Now the starting point for Kontsevich's work on deformation quantization is the following idea. Suppose that there is a good map $H^*(A, A) \to C^*(A, A)$ of "differential graded Lie algebras up to homotopy", compare §2.10. Then we should still be able to use this map to show there is a solution to the structural equation in $C^*(A, A)$. In other words the existence of a solution to the structural equation is homotopy invariant in some suitable sense. These "differential graded Lie algebras up to homotopy" are known as L_∞ algebras and they are algebras over a differential graded operad which is an almost free resolution of the operad governing the structure of Lie algebras. They are described explicitly in [**14**] and we have more to say about the definition in §7.3.

The structural equation

$$dC + \frac{1}{2}[C, C]$$

still makes sense for L_∞ algebras. It is also the case that if two L_∞ algebras are homotopy equivalent as L_∞ algebras then there is a solution to the structural equation in one if and only if there is a solution to the structural equation in the other. The most important step in Kontsevich's work is the proof of following theorem, which is known as Kontsevich's *formality theorem.*

THEOREM 4.5. *$H^*(C^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^n))$ and $C^*(C^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^n))$ are homotopy equivalent as L_∞ algebras.*

Combined with the ideas described above this shows that there is a $*$ product for any Poisson structure on $\mathbb{R}^n$. See Kontsevich's paper [**12**] for more details

5. Some of the general theory of operads

Now we will look more closely at the theory of operads. The basic idea is this. We would like to think of operads as monoids in a monoidal category. Recall that a monoidal category is a category $\mathcal{C}$ equipped with a bifunctor

$$\square : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

which is associative and unital up to natural isomorphisms which satisfy appropriate coherence conditions, see [**13**] for the precise definition. The basic example to keep in mind is the category of vector spaces over a field K with the bifunctor defined by tensor product.

A monoid in a monoidal category $\mathcal{C}$ is an object A in this category equipped with a maps

$$\mu : A\square A \to A, \quad \eta : I \to A$$

where I is the unit for $\square$. These maps are associative and unital in the sense that the following diagram commutes

$$\begin{array}{ccc} A\square A\square A & \xrightarrow{\mu\square 1} & A\square A \\ {\scriptstyle 1\square\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A\square A & \xrightarrow[\mu]{} & A \end{array}$$

and the following composites are both the identity

$$A = I\square A \xrightarrow{\eta\square 1} A\square A \xrightarrow{\mu} A$$

$$A = A\square I \xrightarrow{1\square\eta} A\square A \xrightarrow{\mu} A$$

These diagrams use implicitly the associativity and unital isomorphisms for the bifunctor $\square$. For example, an algebra over a field K is a monoid in the category of vector spaces over K.

We can then use the analogy between monoids in a monoidal category and algebras to generalize constructions with algebras (for example the tensor algebra, the tensor coalgebra, the bar construction and the cobar construction) to monoids in a monoidal category. We will describe the theory for differential graded operads that is operads in the category of chain complexes. This includes graded operads as the special case where the differential is zero. However, the methods we use apply to operads in any symmetric monoidal category, in particular to topological operads.

5.1. The monoidal category of $\mathbb{S}$-spaces.

DEFINITION 5.1. An $\mathbb{S}$-space X is a sequence of differential graded vector spaces $X(k)$, $k = 0, 1, 2, \ldots$, such that the space $X(k)$ is is a representation of the symmetric group Σ_k. A morphism $f : X \to Y$ of $\mathbb{S}$-spaces is a sequence of equivariant maps $f(k) : X(k) \to Y(k)$.

Clearly $\mathbb{S}$-spaces form a category. Following Smirnov, see for example [**14**], we now construct a monoidal operation $\square$ on the category of $\mathbb{S}$-spaces. Let X and Y be $\mathbb{S}$-spaces. For each n we would like to construct a representation $X\square Y(n)$ with the property that given a third $\mathbb{S}$-space Z then a map of $\mathbb{S}$ spaces

$$X\square Y \to Z$$

is uniquely determined by the following data. For each order preserving map $f : \sigma(n) \to \sigma(k)$ we must be given a linear map

$$X(k) \otimes Y(f) \to Z(n),$$

which is equivariant with respect to Σ_f. Here the action of Σ_f on $X(k) \otimes Y(f)$ is its action on $Y(f)$ extended by the identity on $X(k)$. Furthermore, for each order preserving map $f : \sigma(n) \to \sigma(k)$ and each permutation $\pi \in \Sigma_k$ the following diagram must commute

$$\begin{array}{ccc} X(k) \otimes Y(f) & \xrightarrow{\pi \otimes Y(\pi)} & X(k) \otimes Y(\pi_* f) \\ \downarrow & & \downarrow \\ Z(n) & \xrightarrow[Z(f^* \pi)]{} & Z(n) \end{array}$$

We are using the notation of §2.

These two conditions force the definition of $X\square Y(n)$. Let $f : \sigma(n) \to \sigma(k)$ be an order preserving map. Let $\bar{Y}(f)$ be the induced representation

$$\bar{Y}(f) = K(\Sigma_n) \otimes_{\Sigma_f} Y(f).$$

Note that a permutation $\pi \in \Sigma_k$ defines a canonical isomorphism of Σ_n modules

$$\bar{Y}(\pi) : \bar{Y}(f) \to \bar{Y}(\pi_* f).$$

Now form

$$\bar{Y}(n, k) = \sum_{f \in \Delta(n,k)} \bar{Y}(f).$$

The isomorphisms $\bar{Y}(\pi) : \bar{Y}(f) \to \bar{Y}(\pi_* f)$ define an action of Σ_k on $\bar{Y}(n, k)$ and since they are isomorphisms of Σ_n modules we see that $Y(n, k)$ is a (Σ_n, Σ_k)-bimodule. Now form

$$X\square Y(n) = \sum_{k=0}^{n} Y(n, k) \otimes_{\Sigma_k} X(k).$$

It is clear that $X\square Y(n)$ has the universal property described above.

Let I be the $\mathbb{S}$ object defined by

$$I(k) = \begin{cases} K & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

THEOREM 5.2. • *For any* $\mathbb{S}$ *space* X *there are natural isomorphisms*

$$I\square X = X\square I = X.$$

- *If* X, Y *and* Z *are* $\mathbb{S}$ *spaces then there are natural isomorphisms*

$$X\square(Y\square Z) = (X\square Y)\square Z.$$

- *The category of* $\mathbb{S}$*-spaces has the structure of a monoidal category with product* $\square$.

This theorem is, in essence, an elaboration of the associativity isomorphisms

$$A \otimes_G (B \otimes_H C) = (A \otimes_G B) \otimes_H C$$

where G and H are finite groups, A is a right G-module B is a (G, H) bimodule and C is a left bimodule. See [**14**] for details.

If V is a differential graded vector space then V defines an $\mathbb{S}$ object $V[1]$ by

$$V[1](k) = \begin{cases} V & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that

$$V[1]\square V[1] = (V \otimes V)[1].$$

Thus the monoidal structure on $\mathbb{S}$ spaces can be regarded as an extension of the tensor product. More formally, the rule $V \mapsto V[1]$ defines a faithful monoidal functor from the category of differential graded vector spaces to the category of $\mathbb{S}$ spaces.

DEFINITION 5.3. An operad is a monoid in the category of graded $\mathbb{S}$ spaces.

The universal property of the product $\square$ shows that this rather concise definition is indeed the same as the definition given in §1.

Now we formulate the definition of an algebra over an operad in terms of the operation $\square$. Recall that a module over the monoid $\mathfrak{a}$ is an $\mathbb{S}$ object X equipped with a map

$$\mathfrak{a}\square X \to X$$

satisfying the obvious associativity and unital conditions. An $\mathfrak{a}$ module structure on X is determined by a sequence of maps

$$a(k) \otimes X(n_1) \otimes \cdots \otimes X(n_k) \to X(n)$$

where $n = n_1 + \cdots + n_k$. Suppose now that V is a graded vector space. Define a graded $\mathbb{S}$ space $V[0]$ by

$$V[0](k) = \begin{cases} V & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

To make $V[0]$ into a module over the monoid $\mathfrak{a}$ reduces to giving maps

$$\mathfrak{a}(k) \otimes V^{\otimes k} \to V.$$

This immediately suggests how we should define $\mathfrak{a}$ algebras.

DEFINITION 5.4. A differential graded vector space V is an algebra over the operad $\mathfrak{a}$ if the $\mathbb{S}$ space $V[0]$ is a module over the monoid $\mathfrak{a}$.

It is easy to check this is equivalent to the definition given in §1.

One advantage of this approach to the definition of an operad is that it immediately tells us how to define cooperads and coalgebras over a cooperad.

DEFINITION 5.5. A cooperad $\mathfrak{z}$ is a comonoid in the monoidal category of $\mathbb{S}$ spaces. A differential graded vector space V is a coalgebra over $\mathfrak{z}$ if $V[0]$ is a comodule over the comonoid $\mathfrak{z}$.

5.2. The tensor operad and the tensor cooperad. The constructions in this section are the analogues of the constructions of tensor algebras and tensor coalgebras in §3 with differential graded vector spaces replaced by $\mathbb{S}$ spaces and tensor product $\otimes$ replaced by $\square$. First we explain how to construct the tensor operad generated by a graded $\mathbb{S}$-space X.

Let X be a graded $\mathbb{S}$-space. Define $T^k(X)$ inductively as follows

$$T^0(X) = I, \quad T^{k+1}(X) = T^k(X) \square X.$$

The $\mathbb{S}$ space $T(X)$ is the analogue for $\mathbb{S}$ spaces of the k-fold tensor product of a vector space. Indeed, if V is a differential graded vector space and $V[1]$ is the $\mathbb{S}$ space define in the previous section, then

$$T^k(V[1]) = T^k(V)[1]$$

Now define

$$T(X) = \sum_{k=0}^{\infty} T^k(X).$$

The natural isomorphisms

$$T^k(X) \square T^l(X) \to T^{k+l}(X)$$

provided by the monoidal structure of the category of $\mathbb{S}$-spaces define a map of$\mathbb{S}$-spaces

$$T(X) \square T(X) \to T(X)$$

and it is not difficult to check that this map makes $T(X)$ into a monoid in the category of $\mathbb{S}$ spaces. The $\mathbb{S}$ space $T(X)$ equipped with this operad structure will be denoted

$$\mathrm{Op}(X)$$

and will be referred to as the *tensor operad* of X.

Suppose now we have a map of $\mathbb{S}$ spaces

$$f : X \to \mathfrak{a}$$

where $\mathfrak{a}$ is an operad. Then we may use the operad structure of $\mathfrak{a}$ expressed as a map $\mathfrak{a} \square \mathfrak{a} \to \mathfrak{a}$ to construct maps

$$f_k : T^k(X) \to \mathfrak{a}$$

and these maps are the components of a map

$$\mathrm{Op}(X) \to \mathfrak{a}.$$

This is obviously a map of operads. This construction shows that $\mathrm{Op}(X)$ is the *free operad* generated by the graded $\mathbb{S}$ space X.

It is possible to make this construction more explicit. See [**7, 8, 14**] for the details.

Having defined free operads we now give the definition of almost free operads promised in §2.10.

DEFINITION 5.6. A differential graded operad is *almost free* if the underlying graded operad is a free graded operad. A map $\mathfrak{b} \to \mathfrak{a}$ of differential graded operads is an *almost free resolution* of $\mathfrak{a}$ if it induces an isomorphism in homology and $\mathfrak{b}$ is almost free.

As outlined in §2.10 we can use an almost free resolution of $\mathfrak{a}$ to define the notion of an $\mathfrak{a}$ algebra up to homotopy.

We can also define the *tensor cooperad* of by X by setting

$$\mathrm{Coop}(X) = \sum T^k(X)$$

with the cooperad structure defined by the natural isomorphisms

$$T^{p+q}(X) \to T^p(X) \square T^q(X).$$

The details are quite straightforward. The universal property of this tensor cooperad requires some thought about what is meant by a conilpotent cooperad and we will not try to give the details.

A *derivation of an operad* $\mathfrak{a}$ is a map $D : \mathfrak{a} \to \mathfrak{a}$ of graded $\mathbb{S}$ spaces such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{a}\square\mathfrak{a} & \xrightarrow{\mu} & \mathfrak{a} \\ {\scriptstyle D\square 1 + 1\square D}\downarrow & & \downarrow{\scriptstyle D} \\ \mathfrak{a}\square\mathfrak{a} & \xrightarrow[\mu]{} & \mathfrak{a} \end{array}$$

where $\mu : \mathfrak{a}\square\mathfrak{a} \to \mathfrak{a}$ is the structure map. A *coderivation of a cooperad* $\mathfrak{z}$ is a map $L : \mathfrak{z} \to \mathfrak{z}$ of graded $\mathbb{S}$ spaces such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{z} & \xrightarrow{\psi} & \mathfrak{z}\square\mathfrak{z} \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle L\square 1 + 1\square L} \\ \mathfrak{z} & \xrightarrow[\psi]{} & \mathfrak{z}\square\mathfrak{z} \end{array}$$

Note that a differential graded operad is just a graded operad equipped with a derivation d of degree -1 such that $d^2 = 0$ and a differential graded cooperad is graded cooperad equipped with a coderivation of degree -1 and square zero.

We write $\mathrm{Der}(\mathfrak{a})$ for the space of derivations of $\mathfrak{a}$ and $\mathrm{Coder}(\mathfrak{z})$ for the space of coderivations of a cooperad $\mathfrak{z}$. Just as in the case of algebras and coalgebras we can compute the derivations of a tensor operad and a tensor cooperad

$$\mathrm{Der}(\mathrm{Op}(X)) = \mathrm{Hom}(X, \mathrm{Op}(X)),$$
$$\mathrm{Coder}(\mathrm{Coop}(X)) = \mathrm{Hom}(\mathrm{Coop}(X), X).$$

5.3. Bar and cobar constructions. We now discuss the bar construction of a differential graded operad and the cobar construction of a differential graded cooperad. This is a completely straightforward copy of the corresponding constructions of §3 and so we will only outline the details very briefly.

If $\mathfrak{a}$ is a differential graded operad first form $\mathrm{Coop}(\Sigma\mathfrak{a})$. Here $\Sigma\mathfrak{a}$ is the $\mathbb{S}$ space defined by

$$(\Sigma\mathfrak{a})(n) = \Sigma\mathfrak{a}(n).$$

The differential

$$d : \mathfrak{a} \to \mathfrak{a}$$

and the structure map

$$\mu : \mathfrak{a}\Box\mathfrak{a} \to \mathfrak{a}$$

give maps

$$\mathrm{Coop}(\Sigma\mathfrak{a}) \xrightarrow{p_1} \Sigma\mathfrak{a} \longrightarrow \Sigma\mathfrak{a}$$
$$\mathrm{Coop}(\Sigma\mathfrak{a}) \xrightarrow{p_2} \Sigma\mathfrak{a}\Box\Sigma\mathfrak{a} \longrightarrow \Sigma\mathfrak{a}.$$

These maps in turn define coderivations of $\mathrm{Coop}(\Sigma\mathfrak{a})$ and the sum of these coderivations is a coderivation with square zero. The *bar construction* $B(\mathfrak{a})$ on the operad $\mathfrak{a}$ is the differential graded operad with underlying operad $\mathrm{Coop}(\mathfrak{a})$ equipped with this differential.

Similarly we can from the cobar construction$F(\mathfrak{z})$ on the cooperad $\mathfrak{z}$ by mimicking the cobar construction of a coalgebra in the obvious way. Just as in the case of algebras we get a natural map of operads

$$F(B(\mathfrak{a})) \to \mathfrak{a}$$

for any operad $\mathfrak{a}$.

THEOREM 5.7. *This map $F(B(\mathfrak{a})) \to \mathfrak{a}$ is an isomorphism in homology.*

Thus the map $F(B(\mathfrak{a})) \to \mathfrak{a}$ is a resolution of the operad $\mathfrak{a}$ in the sense of previous section. If we forget the differential on $F(B(\mathfrak{a}))$ it is a free graded operad and so $F(B(\mathfrak{a})) \to \mathfrak{a}$ is an almost free resolution of $\mathfrak{a}$. This is the *canonical resolution* of $\mathfrak{a}$. However, even though the canonical resolution is a universal construction it is also a very big construction. The underlying $\mathbb{S}$ module is $T(T(\mathfrak{a}))$. Clearly we would like to find natural constructions of smaller almost free resolutions.

6. A_∞-algebras

In this section we outline the theory of A_∞-algebras or associative algebras up to homotopy. These were originally introduced by Stasheff in [**18**], compare also [**6**]. The objective is to understand what it means to say that a graded vector space V is an associative algebra up to homotopy.

6.1. The definition of an A_∞ algebra. The starting point is the bar construction $B(A)$ of a graded algebra A. As a coalgebra $B(A)$ is the tensor coalgebra $C(\Sigma A)$. The differential L_m is the unique coderivation of $C(\Sigma A)$ defined by the linear map

$$C(\Sigma A) \xrightarrow{p_2} \Sigma A \otimes \Sigma A \xrightarrow{\tilde{m}} \Sigma A.$$

where p_2 is the projection onto the summand $\Sigma A \otimes \Sigma A$ and $\tilde{m}$ is the degree 1 product on ΣA determined by the product $m : A \otimes A \to A$. Note that A is just a graded algebra, so there is no differential in A and the differential in $B(A)$ depends only on the product in A. As in §2.1 the associativity of m gives the formula

$$\tilde{m} \circ (\tilde{m} \otimes 1) = -\tilde{m}(1 \otimes \tilde{m}).$$

From the expression for L_m^2 we see that this is precisely the universal condition which ensures that $L_m^2 = 0$. It is very important to note how the associativity of the product m is encoded in the fact that $L_m^2 = 0$ since this gives us an obvious way to relax the associativity condition.

DEFINITION 6.1. An A_∞ structure on a graded vector space V is a coderivation $L : C(\Sigma V) \to C(\Sigma V)$ of the tensor coalgebra of V such that

- L has degree -1,

- $L^2 = 0$,
- $L(1) = 0$ where $1 \in T^0(\Sigma V) = K$ is the unit in the ground field.

Thus if L is an A_∞ structure on V it follows that $C(\Sigma V)$ with differential L is a counital coaugmented differential graded coalgebra. This suggests the definition of an A_∞ morphism between two A_∞ algebras.

DEFINITION 6.2. Let (V, L_V) and (W, L_W) be two A_∞ algebras. Then an A_∞ morphism $f : (V, L_V) \to (W, L_W)$ is a map of counital coaugmented differential graded coalgebras

$$f : (C(\Sigma V), L_V) \to (C(\Sigma W), L_W).$$

6.2. Some formulas. We will now make these definitions more explicit. If (V, L) is an A_∞ algebra then the coderivation L of $C(\Sigma V)$ is uniquely determined by its components

$$L_j : T^j(\Sigma V) \to \Sigma V, \quad |L_j| = -1.$$

This map L_j is the composite

$$T^j(\Sigma V) \xrightarrow{i_j} C(\Sigma V) \xrightarrow{L} C(\Sigma V) \xrightarrow{p_1} \Sigma V.$$

Since $L(1) = 0$ it follows that

$$L_0 = 0.$$

Now $L^2 = 0$ if and only if all its components $(L^2)_j$ are zero. The components of L^2 are given by the formula

$$(L^2)_j = \sum_{a=1}^{j} L_a \left(\sum_{b=0}^{a} 1^{\otimes b} \otimes L_{j-a+1} \otimes 1^{\otimes a-b-1} \right)$$

For $j = 1, 2, 3$ the equations $(L^2)_j = 0$ become

$$L_1 L_1 = 0$$

$$L_1 L_2 + L_2(L_1 \otimes 1 + 1 \otimes L_1) = 0$$

$$L_1 L_3 + L_2(L_2 \otimes 1 + 1 \otimes L_2) + L_3(L_1 \otimes 1 \otimes 1 + 1 \otimes L_1 \otimes 1 + 1 \otimes 1 \otimes L_1) = 0$$

We now convert the L_j into maps

$$m_j : T^j(V) \to V$$

by defining

$$m_j = \sigma^{-1} L_j T^j(\sigma), \quad |m_j| = j - 2$$

Since $L_0 = 0$ it follows that

$$m_0 = 0.$$

We can now express the identities the L_j must satisfy as identities the m_j must satisfy: these identities have the same form as the identities amongst the L_j but the signs are different. The first three are

$$m_1 m_1 = 0$$

$$m_1 m_2 - m_2(m_1 \otimes 1 + 1 \otimes m_1) = 0$$

$$m_1 m_3 - m_2(m_2 \otimes 1 - 1 \otimes m_2) + m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_3 \otimes 1 + 1 \otimes 1 \otimes m_1) = 0.$$

We can interpret the relations amongst the m_j as follows. The map m_1 has degree -1 and $m_1^2 = 0$. This tells us that m_1 defines a boundary map on V. From now on we will write

$$d = m_1 : V \to V.$$

The degree of m_2 is zero and the second identity is

$$dm_2 = m_2(d \otimes 1) + m_2(1 \otimes d).$$

This tells us that m_2 defines a product of degree 0 on V and d is a derivation of this product. The third identity is

$$dm_3 + m_3(d \otimes 1 \otimes 1 + 1 \otimes d \otimes 1 + 1 \otimes 1 \otimes d) = m_2(m_2 \otimes 1) - m_2(1 \otimes m_2).$$

This formula tells us that m_3 is a chain homotopy from the associator map

$$m_2(m_2 \otimes 1) - m_2(1 \otimes m_2) : V \otimes V \otimes V \to V$$

to 0. So m_2 defines a product on $H_*(V, d)$ and this product is associative. The higher identities are "homotopies between homotopies" or coherence conditions.

We can carry out a similar detailed analysis of a morphism of A_∞ algebras. The morphism f is a map of differential graded coalgebras

$$f : (C(\Sigma V), L_V) \to (C(\Sigma W), L_W).$$

In particular, it is a map of coalgebras and as in §3.3 it is uniquely determined by the maps

$$f_j : T^j(\Sigma V) \to \Sigma W$$

which are the composites

$$T^j(\Sigma V) \xrightarrow{i_j} C(\Sigma V) \xrightarrow{f} C(\Sigma W) \xrightarrow{p_1} \Sigma W.$$

Here i_j is the natural inclusion and p_1 is the natural projection. Since f preserves the counit and the coaugmentation it follows that

$$f_0 = 0.$$

Also f must commute with the differentials, that is

$$L_W f = f L_V.$$

Since L_W, L_V are coderivations and f is a map of coalgebras this is equivalent to

$$p_1 L_W f = p_1 f L_V$$

(think about the dual statement if this seems unclear). In turn, this is equivalent to the identities

$$p_1 L_W f i_n = p_1 f L_V i_n.$$

Now using Lemma 3.4 we see that

$$p_1 L_W f i_n = \sum_k L_{W,k} \sum_{n_1+\cdots+n_k=n} f_{n_1} \otimes \cdots \otimes f_{n_k},$$

and

$$p_1 f L_V i_n = \sum_{a+b+c=n} f_{a+c+1}(1^{\otimes a} \otimes L_{V,b} \otimes 1^{\otimes c}).$$

where $L_{V,i}$ and $L_{W,j}$ are the components of L_V and L_W. Thus the identities amongst the f_j, $L_{V,k}$ and $L_{W,l}$ which correspond to the fact that $f : (C(\Sigma V), L_V) \to (C(\Sigma W), L_W)$ is map of counital coaugmented differential graded coalgebras are

$$\sum_k (L_{W,k} \sum_{n_1+\cdots+n_k=n} f_{n_1} \otimes \cdots \otimes f_{n_k}) = \sum_{a+b+c=n} \phi_{a+c+1}(1^{\otimes a} \otimes L_{V,b} \otimes 1^{\otimes c}).$$

These formulas are in general not very illuminating especially if we realize that they are all written in terms of ΣV and ΣW and to convert them into formulas in terms of V and W involves working through the signs introduced by this process. However

it is worth writing down explicitly the cases $n = 1, 2$. As before we write $m_{V,k}$ and $m_{W,k}$ for the linear maps $T^k(V) \to V$ and $T^k(W) \to W$ determined by the A_∞ structures on V and W and $\phi_k : T^k(V) \to W$ for the linear maps determined by the $f_k : T^k(\Sigma V) \to \Sigma W$.

In the case case $n = 1$ the relation becomes

$$m_{W,1}\phi_1 = \phi_1 m_{V,1}$$

and this tells us that

$$\phi_1 : V \to W$$

is a map of chain complexes. In the case $n = 2$ the relation becomes

$$m_{W,1}\phi_2 + \phi_2(m_{V,1} \otimes 1 + 1 \otimes m_{V,1}) = m_{W,2}(\phi_1 \otimes \phi_1) - \phi_1 m_{V,2}.$$

This tells us that ϕ_2 is a chain homotopy from $m_{W,2}(\phi_1 \otimes \phi_1)$ to $\phi_1 m_{V,2}$, in other words ϕ_1 preserves the product up to homotopy.

6.3. The A_∞ operad. An associative algebra is an A_∞ algebra since the associative product defines a coderviation of $C(\Sigma A)$ with the required properties. Further if V is an A_∞ algebra then $H_*(V, d)$ is an associative algebra. Thus we have the two basic properties of an associative algebra up to homotopy. It is natural to ask how this definition of an A_∞ algebra fits in with the general discussion in §2.10.

In fact, an A_∞ algebra is an algebra over a differential graded operad. We outline the construction of the corresponding operad. Clearly this operad must be generated by the operators m_i of degree $i - 2$ in some sense. Notice also that the operator m_1 is special since it defines the boundary operator on V. So consider the operators m_i with $i \geq 2$. We have not yet built in permutations so we use these m_k with $k \geq 2$ to generate an $\mathbb{S}$ space X in the following way

$$X(k) = \begin{cases} 0 & \text{if } k = 0, 1 \\ K(\Sigma_k) & \text{if } k \geq 2 \end{cases}$$

where he generator of $X(k)$ is m_k. Now if we inspect the relations amongst the m_k imposed by the condition that $L^2 = 0$ we will find that there are no relations involving only the m_k with $k \geq 2$. Thus we are lead to form the free operad $\mathrm{Op}(X)$ generated by the $\mathbb{S}$ space X defined above. Now we use the relations amongst the m_i to define a differential on this free operad. The method is as follows. The relations can all be written in the form

$$dm_k + \sum m_k(1^i \otimes d \otimes 1^{k-i-1}) = A_k$$

where A_k does not involve either $d(= m_1)$ or m_k. This means we can identify A_k with an element of $\mathrm{Op}(X)$. Now define the differential δ on X by setting

$$\delta m_k = A_k.$$

This is extended equivariantly to the $\mathbb{S}$ space X, and then as a derivation of operads to $\mathrm{Op}(X)$. We must check that $\delta^2 = 0$. Finally we must check that if V is an A_∞ algebra then the chain complex (V, d) is indeed an algebra over this operad. What we have outlined in this discussion is the construction of an operad, which we will call A_∞ such that an A_∞ structure on a graded vector space V is the same thing as the structure of an algebra over the operad A_∞ on the chain complex (V, d).

If we take a more sophisticated view of this discussion we see that it is the basis of a proof of the following theorem. To state the theorem, let $\mathfrak{a}$ be an operad

and let $\Lambda\mathfrak{a}$ be the operad characterized by the fact that giving V the structure of an algebra over $\Lambda\mathfrak{a}$ is the same as giving ΣV the structure of an algebra over $\mathfrak{a}$. Explicitly,

$$\Lambda\mathfrak{a}(n) = \Sigma^{1-n}\mathfrak{a}(n) \otimes \mathrm{sign}_n$$

where sign_n is the sign representation of Σ_n. So, for example

$$\Lambda^d L_0 = L_d$$

where L_d is the operad which governs the structure of Lie algebras with Lie bracket of degree d. We can also form $\Lambda\mathfrak{z}$ for a cooperad $\mathfrak{z}$.

THEOREM 6.3. *The operad A_∞ is canonically isomorphic to $F(\Lambda\mathfrak{coass})$, the cobar construction on the coassociative coalgebra cooperad.*

Since every associative algebra is an A_∞-algebra there is a map of operads

$$A_\infty \to \mathfrak{ass}$$

THEOREM 6.4. *This map $A_\infty \to \mathfrak{ass}$ is an almost free resolution of $\mathfrak{ass}$.*

The operad A_∞ is almost free since it is $F(\mathfrak{coass})$, what must be proved is that $H_*(A_\infty) = \mathfrak{ass}$. In the next section we describe the proper generalization of this theorem using the theory of Koszul operads.

So we see that we have a very good understanding of associative algebras up to homotopy. We have constructed a natural almost free resolution A_∞ of the associative algebra operad $\mathfrak{ass}$ and this means that we have good control over associative algebras up to homotopy. We also have a good way of understanding algebras over this resolution. Effectively, algebras over this resolution are differential graded coalgebras with underlying graded coalgebra $C(\Sigma V)$. This allows us to do effective computations in the category of A_∞ algebras. We would like to generalize this construction to other examples, in particular, in view of Kontsevitch's work on deformation quantization, outlined in §4, we would like to know if there is a similarly good theory for Lie algebras up to homotopy and Gerstenhaber algebras up to homotopy.

7. Koszul operads

Koszul operads were introduced by Ginzburg and Kapranov in [**8**]. The idea is that these operads are analogous to Koszul algebras as studied by Priddy in [**16**]. The main reason for singling out Koszul algebras is that the bar complex of such an algebra can be replaced by a much simpler, but equivalent, object. Here we will outline the theory very briefly, see [**8**] for details.

7.1. Generators and relations for operads. Since we have defined free operads we can now define operads by giving generators and relations. Let X be an $\mathbb{S}$ space and let R be a sub $\mathbb{S}$ space of $\mathrm{Op}(X)$. Then an operad $\mathfrak{a}$ is *generated by X with relations R* if it is a pushout in the category of operads of the following form

$$\begin{array}{ccc} \mathrm{Op}(R) & \longrightarrow & \mathrm{Op}(X) \\ \downarrow & & \downarrow \\ I & \longrightarrow & \mathfrak{a} \end{array}$$

Notice that we have to describe $\mathfrak{a}$ as a pushout, rather than a quotient.

Similarly we can define cooperads by giving cogenerators and corelations. Explicitly, let X be an $\mathbb{S}$ space and let Q be a quotient of $\mathrm{Coop}(X)$. Then a cooperad $\mathfrak{z}$ is cogenerated by X with corelations Q if it is the pullback in the category of cooperads of the following form.

$$\begin{array}{ccc} \mathfrak{z} & \longrightarrow & I \\ \downarrow & & \downarrow \\ \mathrm{Coop}(X) & \longrightarrow & \mathrm{Coop}(Q) \end{array}$$

7.2. Quadratic operads and quadratic duals. An operad is *quadratic* if it is generated by an $\mathbb{S}$ object X with $X(j) = 0$ if $j \neq 2$ with relations $R \subset F(X)$ where $R(j) = 0$ if $j \neq 3$.

Given a quadratic operad $\mathfrak{a}$ with generators X and relations R we now construct its *quadratic dual*; this will be a cooperad and will be denoted by $\mathfrak{a}^{\perp}$. To handle gradings and signs properly in this quadratic duality we proceed as follows. First form $\mathrm{Coop}(\Sigma X)$. Now note that $\mathrm{Coop}(\Sigma X)[3] = \Sigma X \square \Sigma X[3]$ and we have the isomorphism

$$\sigma \square \sigma : X \square X \to \Sigma X \square \Sigma X.$$

Using this isomorphism we can form the subspace

$$\sigma \square \sigma R \subset \Sigma X \square \Sigma X[3] = \mathrm{Coop}(\Sigma X)[3]$$

Now define an $\mathbb{S}$ space Q by setting $Q(j) = 0$ if $j \neq 3$ and

$$Q(3) = \mathrm{Coop}(\Sigma X)(3)/\sigma \square \sigma R(3).$$

There is now a natural quotient map of $\mathbb{S}$ spaces

$$\mathrm{Coop}(X) \to Q$$

which is the above quotient in degree 3 and zero in other degrees. Now define $\mathfrak{a}^{\perp}$ to be the cooperad cogenerated by X with corelations Q; this is the *quadratic dual* of $\mathfrak{a}$.

There is a completely analogous construction of a quadratic cooperad $\mathfrak{z}$ and its quadratic dual which is an operad denoted by $\mathfrak{z}^{\perp}$. The detailed definition is left to the reader.

LEMMA 7.1. *If $\mathfrak{a}$ is a quadratic operad then $(\mathfrak{a}^{\perp})^{\perp} = \mathfrak{a}$. If $\mathfrak{z}$ is a quadratic cooperad then $(\mathfrak{z}^{\perp})^{\perp} = \mathfrak{z}$.*

The proof of this lemma follows pretty directly from the definitions. Now we identify the operads $\mathfrak{ass}^{\perp}$, $\mathfrak{com}^{\perp}$ and the quadratic duals of the Lie and Gerstenhaber algebra operads.

To do this let $\mathfrak{a}$ be an operad and let $\Lambda\mathfrak{a}$ be the operad characterized by the fact that giving V the structure of an algebra over $\Lambda\mathfrak{a}$ is the same as giving ΣV the structure of an algebra over $\mathfrak{a}$. Explicitly,

$$\Lambda\mathfrak{a}(n) = \Sigma^{1-n}\mathfrak{a}(n) \otimes \mathrm{sign}_n$$

where sign_n is the sign representation of Σ_n. So, for example

$$\Lambda^d L_0 = L_d$$

where L_d is the operad which governs the structure of Lie algebras with Lie bracket of degree d. From now on we will write L for the operad L_0; this is the operad

which governs the structure of graded Lie algebras, that is Lie algebras with bracket of degree 0 so that $L_d = \Lambda^d L$.

LEMMA 7.2.

$$\mathtt{ass}^! = \Lambda^{-1}\mathtt{ass}^*$$
$$\mathtt{com}^! = \Lambda^{-1}L^*$$
$$L_d^! = \Lambda^{-d-1}\mathtt{com}^*$$
$$G_d^! = \Lambda^{-d-1}G_d^*$$

It is a very good exercise to check these statements.

7.3. Koszul operads. From now on we assume that the ground field K has characteristic zero. We now formulate the Koszul condition on a quadratic differential graded operad $\mathfrak{a}$. There is a canonical map of differential graded cooperads $\mathfrak{a}^! \to B(\mathfrak{a})$, see [**8**] and [**7**] for the definition of this map. The operad $\mathfrak{a}$ is *Koszul* if this map is a quasi-isomorphism, that is it induces an isomorphism of homology operads.

THEOREM 7.3. *The operads* $\mathtt{ass}$, $\mathtt{com}$, L_d *and* G_d *are Koszul.*

See [**8**] for proofs of the first three statements. The fourth is proved in [**7**].

Now suppose $\mathfrak{a}$ is a quadratic operad which is Koszul. We can form the operad

$$F(\mathfrak{a}^!).$$

The map $F(\mathfrak{a}) \to F(B(\mathfrak{a}))$ is a quasi-isomorphism and there is a canonical quasi-isomorphism of $F(B(\mathfrak{a})) \to \mathfrak{a}$. So we see that for a Koszul operad the composition of these maps gives a quasi-isomorphism

$$F(\mathfrak{a}^!) \to \mathfrak{a}.$$

Since $F(\mathfrak{a}^!)$ is an almost free operad we have constructed an almost free resolution of the Koszul operad $\mathfrak{a}$ which we will refer to as the *Koszul resolution.*

THEOREM 7.4.

$$F(\mathtt{ass}^!) = A_\infty, \quad F(L^!) = L_\infty.$$

Here L_∞ is the operad governing the structure of L_∞ algebras. There is a good historical account of the theory of L_∞ algebras in §1.9 of Part 1 of [**14**]. There is a detailed account of the theory of L_∞ algebras in Chapter 3 of Part 2 of [**14**]. So we now see that the operads A_∞ and L_∞ are the Koszul resolutions of the associative algebra operad and the Lie algebra operad.

We conclude with a very brief account of another point of view on this Koszul condition. Suppose we have a quadratic operad $\mathfrak{a}$ with quadratic dual cooperad $\mathfrak{a}^!$. Now let V be a graded vector space. We can now form the $\mathfrak{a}^!$ coalgebra

$$C_{\mathfrak{a}^!}(V)$$

which is the analogue in the category of $\mathfrak{a}^!$ coalgebras of the tensor coalgebra $C(V)$. Explicitly, this object $C_{\mathfrak{a}^!}(V)$ comes equipped with a map $C_{\mathfrak{a}^!}(V) \to V$ which induces an isomorphism

$$\mathrm{Hom}_{\mathfrak{a}^!}(C, C_{\mathfrak{a}^!}(V)) \to \mathrm{Hom}_K(C, V).$$

Here C is an $\mathfrak{a}^!$ coalgebra which satisfies the appropriate version of the conilpotence condition and $\mathrm{Hom}_{\mathfrak{a}^!}$ means homomorphisms of $\mathfrak{a}^!$ coalgebra respecting appropriate counits and coaugmentations and so on. Now, as in §6, an $\mathfrak{a}$ algebra structure

on V defines in a canonical way a coderivation of the $\mathfrak{a}^{\perp}$ coalgebra $C_{\mathfrak{a}^{\perp}}(V)$ with degree -1 and square zero. Thus we are naturally led to define an $\mathfrak{a}_{\infty}$ *structure* on V to be a coderivation of the $\mathfrak{a}^{\perp}$ coalgebra $C_{\mathfrak{a}^{\perp}}(V)$ with degree -1 and square zero.

The first result is that V is an $\mathfrak{a}_{\infty}$ algebra if and only if V is an algebra over the operad $F(\mathfrak{a}^{\perp})$. Since $\mathfrak{a}$ algebras have a canonical $\mathfrak{a}_{\infty}$ structure, there is a map of operads

$$\mathfrak{a}_{\infty} \to \mathfrak{a}.$$

Now, in these terms, $\mathfrak{a}$ is Koszul if and only if this canonical map is a quasi-isomorphism.

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MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY CV4 7AL, UNITED KINGDOM

E-mail address: jdsj@warwick.ac.uk

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Graded manifolds and Drinfeld doubles for Lie bialgebroids

Theodore Voronov

ABSTRACT. We define *graded manifolds* as a version of supermanifolds endowed with an extra $\mathbb{Z}$-grading in the structure sheaf, called *weight* (not linked with parity). Examples are ordinary supermanifolds, vector bundles, double vector bundles (in particular, iterated constructions like TTM), etc. I give a construction of *doubles* for *graded QS-* and *graded QP-manifolds* (graded manifolds endowed with a homological vector field and a Schouten/Poisson bracket). Relation is explained with Drinfeld's Lie bialgebras and their doubles. Graded QS-manifolds can be considered, roughly, as "generalized Lie bialgebroids". The double for them is closely related with the analog of Drinfeld's double for Lie bialgebroids recently suggested by Roytenberg. Lie bialgebroids as a generalization of Lie bialgebras, over some base manifold, were defined by Mackenzie and P. Xu. Graded QP-manifolds give an odd version for all this, in particular, they contain "odd analogs" for Lie bialgebras, Manin triples, and Drinfeld's double.

Introduction

Analog for Lie bialgebroids of Drinfeld's classical double is a puzzle. Various constructions were suggested (Mackenzie, Liu–Weinstein–Xu and Roytenberg), all different and all giving something which is not a Lie bialgebroid, even not a Lie algebroid. Recall that the Drinfeld double of a Lie bialgebra is again a Lie bialgebra (with nice properties). In Roytenberg's thesis [**22**] it was suggested to consider the supermanifold $T^*(\Pi E)$ (with a particular natural structure) as the "right" notion of the double for a Lie bialgebroid $E \to X$ over a base X. (Close ideas are due to A. Vaintrob [**26**].)

In this paper we define a class of supermanifolds which, in particular, generalizes the structure of Lie bialgebroids. We prove that an analog of Roytenberg's construction carries over for them. Namely, we consider a "graded manifold" (see below) M endowed with a homological vector field $\hat{Q}$ of weight q and a compatible Schouten bracket of weight s, and prove that on the space of the cotangent bundle T^*M there is a naturally defined homological field $\hat{Q}_D$ and it is possible to introduce weight for T^*M in such a way that $\hat{Q}_D$ will be again of the weight q (same

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as original $\hat{Q}$), and certain natural properties hold. The space $DM = T^*M$ with such a structure is called the *double* of M. The double DM so defined inherits half the original structure of M, a homological field. Using a linear connection on M, it is possible to define on DM an "almost" Schouten bracket as well. I show that this exactly recovers the whole structure of the Drinfeld double when restricted to Lie bialgebras. (For a nonlinear M, the bracket on DM does not, in general, obey the Jacobi identity.) By considering supermanifolds with a Poisson rather than Schouten bracket, we obtain an odd analog of this theory. We introduce and study *odd Lie bialgebras* and *odd double* for them. In particular, we construct an odd Lie bialgebra structure in the Lie superalgebra $\mathfrak{q}(n)$. (A quantization of this example should give a quantum supergroup $GQ_\varepsilon(n)$ with an odd quantum parameter ε.)

This work is based on two simple ideas.

The first is a key notion of a graded manifold. Probably, the concept is "in the air". Needless to argue for the usefulness of $\mathbb{Z}$-grading in algebraic questions. Unfortunately, there is an often confusion with $\mathbb{Z}_2$-grading responsible for the sign rule. Of course, in many examples the $\mathbb{Z}_2$-grading is induced by some underlying $\mathbb{Z}$-grading, but in other examples not, and a natural $\mathbb{Z}$-grading can have nothing to do with the sign rule (example: degree for polynomials). We introduce *graded manifolds* as a certain version of supermanifolds with a $\mathbb{Z}$-grading in the structure sheaf **independent of parity**. We call it *weight*. This structure embraces many examples and carries over to natural constructions. (The term "graded manifolds" was used in some early works on supermanifolds instead of the latter term. Our usage has no relation with this.)

The second is a "geometrization" of the concept of a Lie algebra, in the following sense. (Everywhere when we say "Lie algebra" we mean "Lie superalgebra".) This is the answer to a simple question: to which structure in the algebra of functions corresponds the Lie bracket? Consider simultaneously the vector space $\mathfrak{g}$ with all its "neighbors": $\mathfrak{g}^*$, $\Pi\mathfrak{g}$, $\Pi\mathfrak{g}^*$. On $\mathfrak{g}^*$ we get a linear Poisson bracket (the Lie–Berezin–Kirillov bracket). On $\Pi\mathfrak{g}^*$ we get a linear odd Poisson (=Schouten, Gerstenhaber) bracket, which we call the Lie–Schouten bracket. On $\Pi\mathfrak{g}$ we get a quadratic homological vector field $Q = \pm(1/2)c_{ij}^k \xi^j \xi^i \partial/\partial\xi^k$. Hence, Poisson brackets, Schouten brackets and homological vector fields are all, equally, extensions of a Lie algebra structure. We call supermanifolds with such structures *P-manifolds*, *S-manifolds* and *Q-manifolds*, respectively [1]. The next question is, what is a "bi-" structure from this viewpoint. Shortly, it is a Lie algebra structure on two neighbors with a compatibility condition. We get the following list: $(\mathfrak{g}, \mathfrak{g}^*)$, which is a Lie bialgebra in Drinfeld's sense, $(\mathfrak{g}, \Pi\mathfrak{g}^*)$, which is the odd analog of Drinfeld's notion (can be non-trivial only for superalgebras), and $(\mathfrak{g}, \Pi\mathfrak{g})$. Their geometric versions are, respectively: *QS-manifolds* (compatible Q- and S-structures), *QP-manifolds* (compatible Q- and P-structures), and *PS-manifolds* (two compatible brackets of the opposite parity; structures of this kind have very interesting geometry, their study was initiated in [**7**]). Notice, that for a "bi-" Lie structure of each type there is just one geometric manifestation (one type of structure, though it can be realized on different spaces, e.g., the QS-structures on $\Pi\mathfrak{g}$ and $\Pi\mathfrak{g}^*$ for a Drinfeld

[1] Curiously, the most difficult is to tell what structure in the algebra of functions on $\mathfrak{g}$ itself corresponds to the Lie bracket on $\mathfrak{g}$. An analogous question is, what natural operator exists on functions on TM, i.e., on Lagrangians. Partial answer, of course, is: the Euler–Lagrange equations. Their analog for Lie algebras and Lie algebroids was considered by A. Weinstein [**35**].

Lie bialgebra $\mathfrak{g}$), compared to the three different manifestations of a "single" Lie bracket.

Combining these two ideas, we say that instead of conventional Lie (bi)algebroids one should deal with graded manifolds with QS- and QP-structure and keep track of weights. In particular, this allows to develop a nice framework for doubles. As a byproduct, we naturally encounter odd analogs for Lie bialgebras and bialgebroids.

The paper is organized as follows:

In Section 1 we review the necessary facts concerning Poisson and Schouten algebras. In particular, it is shown how an arbitrary Poisson structure is expressed via the canonical Schouten structure on the anticotangent bundle, and an arbitrary Schouten structure is expressed via the canonical Poisson bracket on the cotangent bundle. We see how even and odd brackets intertwine.

In Section 2 we give a geometric description of Drinfeld's Lie bialgebras, in the language of supermanifolds, and we explain how the classical Drinfeld's double for $\mathfrak{g}$ is equivalent to a natural construction of a QS-structure on $T^*\Pi\mathfrak{g} \cong \Pi(\mathfrak{g} \oplus \mathfrak{g}^*)$. This also leads us to a new notion of an *odd Lie bialgebra*, which we define in terms of a QP-structure on $\Pi\mathfrak{g}$.

In Section 3 we review the "super" approach to Lie bialgebroids and doubles. It is a generalization of the "super" description of Lie bialgebras and their Drinfeld doubles given in Section 2. We describe Roytenberg's "Drinfeld double" of a Lie bialgebroid. It is no longer a Lie algebroid. Actually, it is an example of a graded manifold (which we define in the next section).

In Section 4 we introduce graded manifolds as supermanifolds with a "weight" in the algebra of functions, and show how it carries over to natural bundles and the canonical brackets. In particular, we show that the natural bundles like TM are bi-graded. This gives important flexibility for the definition of a "total" weight.

In Section 5 we consider graded QS- and QP-manifolds. The former should be regarded as generalized Lie (anti)bialgebroids. The latter are a generalization of odd Lie bialgebras defined in Section 2. For both we give a construction of a *double* as a graded Q-manifold. For a graded QS-manifold ΠE, where E is a Lie bialgebroid, this recovers Roytenberg's construction. We show how with the help of a connection it is possible to construct an "almost" bracket in the double.

In Section 6 we consider odd Lie bialgebras and the *odd double* in detail. We consider as example the matrix superalgebra $\mathfrak{q}(n)$ and show that it has a natural odd bialgebra structure. As a technical tool to define this structure, we introduce a "relative" version of doubles, in the situation when there is an action of some Lie superalgebra. (This models certain triangular decompositions of superalgebras.)

In Appendix we give a proof of a theorem of Mackenzie and Xu about cotangent bundles for dual vector bundles. We also suggest and prove its odd analog.

Note about usage. All our constructions are "superized", i.e., we automatically work with supermanifolds, superalgebras, super Lie (bi)algebroids, etc., from the beginning. "Algebras" and "manifolds" stand for "superalgebras" and "supermanifolds" throughout the text unless otherwise required for clarity. Π stands for the parity reversion functor. We use the prefix "anti" for *opposite* vector spaces and vector bundles (spaces and bundles with reversed parity). For example, *anticotangent bundle* means ΠT^*M, *Lie antialgebra* means the space $\Pi\mathfrak{g}$ for a Lie algebra $\mathfrak{g}$, an *antivector* (for a vector space V) means an element of ΠV.

1. Poisson and Schouten algebras

For the sake of completeness, we provide some definitions and facts. All vector spaces are assumed $\mathbb{Z}_2$-graded, with tilde denoting parity of a homogeneous element. Commutators, commutativity, derivations, linearity, etc., are understood in the $\mathbb{Z}_2$-graded sense. The numbers 0, 1 denoting the grading are residues modulo 2.

DEFINITION 1.1. A *Poisson algebra* is a vector space $A = A_0 \oplus A_1$ with a bilinear associative multiplication $(a, b) \mapsto ab$ such that $A_i A_j \subset A_{i+j}$ and an even bilinear operation denoted $\{\ ,\ \}$ (i.e., $\{A_i, A_j\} \subset A_{i+j}$) satisfying

$$\{a, b\} = -(-1)^{\tilde{a}\tilde{b}}\{b, a\} \tag{1.1}$$

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{\tilde{a}\tilde{b}}\{b, \{a, c\}\} \tag{1.2}$$

$$\{a, bc\} = \{a, b\}c + (-1)^{\tilde{a}\tilde{b}}b\{a, c\} \tag{1.3}$$

for every homogeneous elements $a, b, c \in A$. The operation $\{\ ,\ \}$ is called *Poisson bracket.*

DEFINITION 1.2. A *Schouten algebra* is a vector space $A = A_0 \oplus A_1$ with a bilinear associative multiplication $(a, b) \mapsto ab$ such that $A_i A_j \subset A_{i+j}$ and an odd bilinear operation denoted $\{\ ,\ \}$ (i.e., $\{A_i, A_j\} \subset A_{i+j+1}$) satisfying

$$\{a, b\} = -(-1)^{(\tilde{a}+1)(\tilde{b}+1)}\{b, a\} \tag{1.4}$$

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(\tilde{a}+1)(\tilde{b}+1)}\{b, \{a, c\}\} \tag{1.5}$$

$$\{a, bc\} = \{a, b\}c + (-1)^{(\tilde{a}+1)\tilde{b}}b\{a, c\} \tag{1.6}$$

for every homogeneous elements $a, b, c \in A$. The operation $\{\ ,\ \}$ is called *Schouten bracket.*

The difference between these two notions is obvious. A Poisson algebra is, in particular, a Lie (super)algebra, and the identity (1.3) tells that for every $a \in A$, the operator $\operatorname{ad} a = \{a,\ \}$ acts as a derivation of the associative product. The operator $\operatorname{ad} a$ has the same parity as a. Similarly, a Schouten algebra is, in particular, a Lie superalgebra, but w.r.t. the reversed parity. The operator $\operatorname{ad} a := (-1)^{\tilde{a}+1}\{a,\ \}$ in a Schouten algebra is a derivation of the associative product of parity opposite to that of a. No shift of parity can turn (1.6) into (1.3).

Other names for Schouten algebras are *odd Poisson algebras* and *Gerstenhaber algebras.* Schouten bracket is also known as *odd bracket* and *antibracket.*

REMARK 1.1. Schouten algebras became very popular recently, in connection with their use in Batalin–Vilkovisky (BV) formalism in quantum field theory [**1**],[**2**] and in deformation quantization (see, e.g., [**8**]). The name "Gerstenhaber algebras" is motivated by the corresponding structure discovered by Gerstenhaber [**4**] in the cohomology of an associative algebra. The name "Schouten algebra" comes from the canonical Schouten bracket of multivector fields [**23**], see below. This bracket is completely parallel with the canonical Poisson bracket on T^*M, so the choice of "Schouten algebras" for the odd counterpart of "Poisson algebras" seems natural. For a manifold provided with a volume element there is another operation on multivector fields, the divergence. It satisfies the relation

$$\delta(PQ) = \delta P\, Q + (-1)^{\tilde{P}} P\, \delta Q + (-1)^{\tilde{P}+1}\{P, Q\}, \tag{1.7}$$

thus connecting Schouten bracket and multiplication. Schouten algebras endowed with a differential satisfying (1.7) received the name *Batalin–Vilkovisky algebras* (BV–algebras), because of their role in the BV–formalism.

DEFINITION 1.3. A (super)manifold with a Poisson (Schouten) structure in the algebra of functions is called a *Poisson* (resp., *Schouten* or *odd Poisson*) *manifold.*

Basic examples: T^*M and the canonical Poisson bracket on functions on T^*M (Hamiltonians on M), ΠT^*M and the canonical Schouten bracket on functions on ΠT^*M (multivector fields on M). They solve two universal problems: for Lie homomorphisms $\operatorname{Vect} M \to A$ to commutative Poisson algebras "over $C^\infty(M)$" and for odd Lie homomorphisms $\operatorname{Vect} M \to A$ to commutative Schouten algebras "over $C^\infty(M)$" (precise formulation requires the notions of Lie–Rinehart algebras or Lie algebroids, see below). In the sequel we will use the following notation for the homomorphisms from vector fields to Hamiltonians and to multivector fields: $p\colon X \mapsto p_X$ and $\theta\colon X \mapsto \theta_X$, respectively. (Notice that the second map is odd.)

A purely algebraic variant is given by the Poisson bracket on $C^\infty(\mathfrak{g}^*)$,

$$\{x_i, x_j\} = c_{ij}^k x_k, \tag{1.8}$$

for a Lie algebra $\mathfrak{g}$, the *Lie–Poisson(–Berezin–Kirillov) bracket*, and by the Schouten bracket on $C^\infty(\Pi\mathfrak{g}^*)$,

$$\{\xi_i, \xi_j\} = c_{ij}^k \xi_k, \tag{1.9}$$

(where the coordinates ξ_i have parity opposite to that of the respective x_i), the *Lie–Schouten bracket*, as we shall call it. They solve universal problems for even/odd bracket-preserving maps $\mathfrak{g} \to A$ to commutative Poisson/Schouten algebras respectively. (For the even case this is directly related with the moment map for Hamiltonian actions of $\mathfrak{g}$.)

Examples above are united in the concept of a Lie algebroid (see [**21, 18**]). A *Lie algebroid* is a vector bundle $E \to M$ over a manifold M endowed with a vector bundle map $a\colon E \to TM$ (called the *anchor*) and a Lie algebra structure on the space of sections $C^\infty(M, E)$ satisfying

$$[u, fv] = a(u)f\,v + (-1)^{\tilde{u}\tilde{f}} f[u, v] \tag{1.10}$$

$$a([u, v]) = [a(u), a(v)] \tag{1.11}$$

for all $u, v \in C^\infty(M, E)$ and all $f \in C^\infty(M)$. Here we identify vector fields on M with derivations of the algebra $C^\infty(M)$. Likewise, for an arbitrary (commutative associative) algebra A, a *Lie pseudoalgebra* or a *Lie–Rinehart algebra* over A is a module L over A with a Lie algebra structure and a homomorphism of modules $a\colon L \to \operatorname{Der} A$ satisfying (1.10),(1.11) (see [**18**]).

It is not difficult to see that every Lie algebroid (or every Lie–Rinehart algebra) canonically produces two algebras, a Poisson algebra and a Schouten algebra. The Poisson algebra associated with a Lie algebroid E is $C^\infty(E^*)$. The Schouten algebra associated with E is $C^\infty(\Pi E^*)$. Axioms of Poisson/Schouten algebra for $C^\infty(E^*)$, $C^\infty(\Pi E^*)$ are equivalent to E being a Lie algebroid. These algebras are universal in a certain precise sense, and their construction unites the constructions of the canonical Poisson bracket in $C^\infty(T^*M)$ and the canonical Schouten bracket in $C^\infty(\Pi T^*M)$ from the commutator of vector fields with the constructions of the Lie–Poisson bracket in $C^\infty(\mathfrak{g}^*)$ and the Lie–Schouten bracket in $C^\infty(\Pi\mathfrak{g}^*)$ from the bracket in $\mathfrak{g}$.

Most often the associative multiplication in the definition of Poisson/Schouten algebras is assumed to be commutative. There are natural examples where it is not so. The commutator bracket $[a,b] = ab - (-1)^{\tilde{a}\tilde{b}}ba$ obviously satisfies the Leibniz identity (1.3) and thus makes every associative algebra into a (noncommutative) Poisson algebra. Let us reserve the notation $[\ ,\ ]$ for the commutator bracket and denote a given arbitrary Poisson bracket by $\{\ ,\ \}$.

LEMMA 1.1 ([**32**]). *In every Poisson algebra the following identity holds:*

$$[a,b]\{c,d\} = \{a,b\}[c,d], \tag{1.12}$$

for arbitrary elements a, b, c, d.

Obviously, (1.12) degenerates in two special cases, when the algebra is commutative and when the bracket coincides with the commutator, as in the example above.

Consider a Lie algebra $\mathfrak{g}$. Let us denote the Lie brackets in $\mathfrak{g}$ by $\{\ ,\ \}$. One may ask about bracket-preserving maps $\mathfrak{g} \to A$ to Poisson algebras without conditions of commutativity.

THEOREM 1.1 ([**32**]). *A non-commutative Poisson algebra that is a universal range for such maps (the "Poisson envelope" of a Lie algebra $\mathfrak{g}$) is given by $E(\mathfrak{g}) := T(\mathfrak{g})/I$ where the ideal I in the free tensor algebra is generated by the elements of the form*

$$(a \otimes b - (-1)^{\tilde{a}\tilde{b}} b \otimes a) \otimes \{c,d\} - \{a,b\} \otimes (c \otimes d - (-1)^{\tilde{c}\tilde{d}} d \otimes c), \tag{1.13}$$

for all $a, b, c, d \in \mathfrak{g}$.

The algebra $C^\infty_{pol}(\mathfrak{g}^*)$ and the universal enveloping algebra $U(\mathfrak{g})$ both are quotients of $E(\mathfrak{g})$. The algebra $E(\mathfrak{g})$ can be viewed as a non-commutative analog of the Kirillov–Kostant–Sourieau moment space $\mathfrak{g}^*$. For details see [**32**].

On a manifold M, a non-degenerate bracket is specified by a symplectic 2-form $\omega \in \Omega^2(M)$, even for the Poisson case and odd for the Schouten case. The correspondence is given by the formulas: $i_{X_f}\omega = -(-1)^{s\tilde{f}}df$, $\{f,g\} = (-1)^{s\tilde{f}}X_f g$, X_f being the Hamiltonian vector field corresponding to a function f, where $s = 0$ for an even bracket and $s = 1$ for an odd bracket.

Consider the general situation. From (1.3),(1.1),(1.6),(1.4) follows that a Poisson as well as a Schouten bracket is a bi-derivation of functions. Hence a bracket is specified by a contravariant tensor field of rank 2. From antisymmetry follows that in the Poisson case it should be an even bivector field. What about the Schouten case? Contrary to what one might think at the first moment, an odd bracket is not related with odd bivector fields. In fact, a Schouten bracket on M is specified by a fiberwise quadratic odd Hamiltonian function[2]. Given an even bivector field $P \in C^\infty(\Pi T^*M)$, a Poisson bracket $\{f,g\}_P$ of functions f and g is defined by the coordinate-free formula

$$\{f,g\}_P := \{f,\{P,g\}\} \tag{1.14}$$

where the brackets at the r.h.s. are the canonical Schouten brackets of multivector fields. The Jacobi identity for $\{\ ,\ \}_P$ is equivalent to the identity $\{P,P\} = 0$. Likewise, given an odd quadratic Hamiltonian $S \in C^\infty(T^*M)$, it defines a Schouten structure $\{\ ,\ \}_S$ by the similar formula

$$\{f,g\}_S := \{f,\{S,g\}\} \tag{1.15}$$

where at the r.h.s. there are the canonical Poisson brackets, and the Jacobi identity for $\{\ ,\ \}_S$ is equivalent to the identity $\{S,S\} = 0$. The interrelation between Poisson

[2]In a non-degenerate situation, brackets of both types are specified by 2-forms. Noteworthy, the inverse matrix for the components of such a form possesses different symmetry properties depending on its parity, corresponding, up to certain sign factors, to the components of a bivector for an even form and to the coefficients of a quadratic function for an odd form.

and Schouten structures is summarized in the table below (see also Appendix B to [**29**]). Notice that $\{f,\{P,g\}\} = \{\{f,P\},g\}$, and $\{f,\{S,g\}\} = \{\{f,S\},g\}$.

A Poisson manifold M	A Schouten manifold M
$\{f,g\}_{P,S} = (-1)^{\tilde{a}(\tilde{f}+1)}\frac{\partial f}{\partial x^a}\{x^a,x^b\}_{P,S}\frac{\partial g}{\partial x^b}$ (same coordinate formula for even and odd bracket)	
$\{x^a,x^b\}_P = (-1)^{\tilde{a}}\,P^{ab}$,	$\{x^a,x^b\}_S = -(-1)^{\tilde{a}}\,S^{ab}$,
P^{ab} the Poisson tensor (bivector field)	S^{ab} the Schouten tensor (Hamiltonian)
$P^{ab} = (-1)^{(\tilde{a}+1)(\tilde{b}+1)}P^{ba}$	$S^{ab} = (-1)^{\tilde{a}\tilde{b}}S^{ba}$
$\widetilde{P^{ab}} = \tilde{a}+\tilde{b}$	$\widetilde{S^{ab}} = \tilde{a}+\tilde{b}+1$
$P = \frac{1}{2}P^{ab}(x)x_b^*x_a^* \in C^\infty(\Pi T^*M)$	$S = \frac{1}{2}S^{ab}(x)p_bp_a \in C^\infty(T^*M)$
Jacobi for $\{\ ,\ \}_P \Leftrightarrow \{P,P\} = 0$	Jacobi for $\{\ ,\ \}_S \Leftrightarrow \{S,S\} = 0$
Explicit formula:	Explicit formula:
$\{f,g\}_P = \{f,\{P,g\}\}$	$\{f,g\}_S = \{f,\{S,g\}\}$
Here $\{\ ,\ \}$ is the canonical Schouten bracket on ΠT^*M. P is even.	Here $\{\ ,\ \}$ is the canonical Poisson bracket on T^*M. S is odd.

(Notation: x^a stand for local coordinates on M, p_a for the induced coordinates in the cotangent space, x_a^* for the induced coordinates in the anticotangent space. Here $\tilde{x}^a = \tilde{a} = \tilde{p}_a$, $\tilde{x}_a^* = \tilde{a}+1$. The transformation law for p_a and x_a^* is the same.)

We shall later need the coordinate formulas for the canonical brackets.

On T^*M the canonical even symplectic form is $dp_adx^a = d(dx^ap_a) = d(p_adx^a)$. The corresponding canonical Poisson bracket:

$$\{f,g\} = (-1)^{\tilde{a}(\tilde{f}+1)}\frac{\partial f}{\partial p_a}\frac{\partial g}{\partial x^a} - (-1)^{\tilde{a}\tilde{f}}\frac{\partial f}{\partial x^a}\frac{\partial g}{\partial p_a}, \tag{1.16}$$

where $f,g \in C^\infty(T^*M)$, i.e., Hamiltonians on M. In particular, $\{p_a,x^b\} = \delta_a^b = -(-1)^{\tilde{a}}\{x^b,p_a\}$.

On ΠT^*M the canonical odd symplectic form is $(-1)^{\tilde{a}+1}dx_a^*dx^a = d(dx^ax_a^*) = d((-1)^{\tilde{a}+1}x_a^*dx^a)$. The corresponding canonical Schouten bracket:

$$\{f,g\} = (-1)^{(\tilde{a}+1)(\tilde{f}+1)}\frac{\partial f}{\partial x_a^*}\frac{\partial g}{\partial x^a} - (-1)^{\tilde{a}(\tilde{f}+1)}\frac{\partial f}{\partial x^a}\frac{\partial g}{\partial x_a^*}, \tag{1.17}$$

where $f,g \in C^\infty(\Pi T^*M)$ (multivector fields on M). In particular, $\{x_a^*,x^b\} = \delta_a^b = -\{x^b,x_a^*\}$.

The properties of the brackets (1.14) and (1.15) are derived from the properties of the canonical brackets. Crucial is that the functions on M pulled-back to T^*M or ΠT^*M have vanishing canonical brackets. In particular, this implies antisymmetry of a new bracket. In such description it is clear that this construction can be put into an abstract setting and generalized (in particular, iterated). It goes as follows.

DEFINITION 1.4 (see [**11**],[**12**]). A *Loday algebra* of parity $s = 0,1$, is a vector space L endowed with a bilinear operation of parity s (a *Loday bracket*) satisfying

the condition

$$[a,[b,c]] = [[a,b]c] + (-1)^{(\tilde{a}+s)(\tilde{b}+s)}[b,[a,c]]. \tag{1.18}$$

In other words, a Loday algebra is a crippled Lie superalgebra without the antisymmetry condition and where the Jacobi identity remains in the form of a one-sided "Leibniz identity" for the bracket. (Loday [**16**] introduced these algebras under the name of "Leibniz algebras".)

Consider an odd derivation D of a Loday algebra L. Introduce a new operation $[a,b]_D := (-1)^{\tilde{a}}[Da,b]$. Obviously, it is a bilinear operation of parity $s+1$.

THEOREM 1.2 ([**11**]). *If $D^2 = 0$, then the bracket $[\ ,\ ]_D$ defines a new structure of a Loday algebra on the space L, of parity $s+1$.*

PROOF. Consider

$$[a,[b,c]_D]_D = (-1)^{\tilde{a}+\tilde{b}}[Da,[Db,c]] =$$
$$(-1)^{\tilde{a}+\tilde{b}}\Big([[Da,Db],c] + (-1)^{(\tilde{a}+s+1)(\tilde{b}+s+1)}[Db,[Da,c]]\Big) =$$
$$(-1)^{\tilde{a}+\tilde{b}}\Big((-1)^{\tilde{a}+s+1}[D[Da,b],c] + (-1)^{(\tilde{a}+s+1)(\tilde{b}+s+1)}[Db,[Da,c]]\Big),$$

because $D[Da,b] = [D^2a,b] + (-1)^{\tilde{a}+s+1}[Da,Db] = -1)^{\tilde{a}+s+1}[Da,Db]$, due to $D^2 = 0$. Thus it equals (we continue)

$$(-1)^{\tilde{a}+\tilde{b}+s+1}[D(-1)^{\tilde{a}}[Da,b],c] + (-1)^{(\tilde{a}+s+1)(\tilde{b}+s+1)}(-1)^{\tilde{b}}[Db,(-1)^{\tilde{a}}[Da,c]] =$$
$$[[a,b]_D,c]_D + (-1)^{(\tilde{a}+s+1)(\tilde{b}+s+1)}[b,[a,c]_D]_D.$$

□

The new bracket (of the opposite parity) $[\ ,\ ]_D$ is called a *derived bracket*. Suppose L is a Lie algebra of parity s (cf. in [**32**]), i.e., there is the antisymmetry condition $[a,b] = -(-1)^{(\tilde{a}+s)(\tilde{b}+s)}[b,a]$ in addition to (1.18). Without loss of generality set $s = 0$. Then $[a,b]_D + (-1)^{(\tilde{a}+1)(\tilde{b}+1)}[b,a]_D = (-1)^{\tilde{a}}[Da,b] + (-1)^{\tilde{a}\tilde{b}+\tilde{a}+1}[Db,a] = (-1)^{\tilde{a}}[Da,b] + [a,Db] = (-1)^{\tilde{a}}D[a,b]$. Thus on a subspace where $D[a,b] = 0$ (or on a certain quotient) the derived bracket gives a genuine Lie bracket of parity $s+1$. (See [**11**] for details and generalizations. Without Loday algebras, derived brackets were also considered in [**30, 31**].)

REMARK 1.2. The derived bracket construction is seen also in the formulas of the Cartan calculus, where $[d,i_u] = L_u$ and $[L_u,i_v] = (-1)^{\tilde{u}}i_{[u,v]}$ imply $i_{[u,v]} = [i_u,[d,i_v]]$, which is a coordinate-free relation of the structure equation for the commutator of vector fields with that for d.

The construction of a Poisson/Schouten bracket from the canonical brackets fits into this abstract model, with $D = \operatorname{ad} P$ or $D = \operatorname{ad} S$, respectively. If we consider an arbitrary even multivector field P satisfying $\{P,P\} = 0$, rather than a bivector field, then the subalgebra $C^\infty(M) \subset C^\infty(\Pi T^*M)$ ceases to be closed under the new bracket. Instead, we arrive at a "Loday–Poisson" structure on the whole of $C^\infty(\Pi T^*M)$. Same holds for the Schouten case, if the restriction that the Hamiltonian S is quadratic in p_a is dropped.

REMARK 1.3. In the algebra of functions on a Schouten manifold M endowed with a volume form there is a natural "odd Laplacian"

$$\Delta f := \frac{1}{2} \operatorname{div} X_f, \tag{1.19}$$

which makes it into a Batalin–Vilkovisky algebra. The formula (1.19) is due to Khudaverdian [**7**]. It provides an invariant geometric setup for the Δ-operator of the BV–formalism (see also [**5**],[**6**]). Notice an analogy with the modular field on Poisson manifolds (see [**34**]). For ΠT^*M considered with the volume form $\boldsymbol{\rho}^2$, where $\boldsymbol{\rho}$ is a volume form on M, the operator (1.19) coincides with the usual divergence of multivector fields δ.

2. "Bi-" structures and the Drinfeld double in a supermanifold framework

Recall the following definition. For simplicity let us first consider the purely even situation. Later we will give a different approach that automatically handles the general case.

DEFINITION 2.1 (Drinfeld). A *Lie bialgebra* is a Lie algebra $\mathfrak{g}$ such that the dual space $\mathfrak{g}^*$ is also a Lie algebra and the two structures are compatible in the sense that

$$\delta\colon \mathfrak{g} \to \wedge^2 \mathfrak{g} \tag{2.1}$$

is a 1-cocycle of $\mathfrak{g}$ w.r.t. $\wedge^2 \operatorname{ad}$, where $\delta^*\colon \wedge^2 \mathfrak{g}^* \to \mathfrak{g}^*$ is the bracket on $\mathfrak{g}^*$.

If e_i is a basis of $\mathfrak{g}$, e^i the dual basis of $\mathfrak{g}^*$, then the definition of a bialgebra is equivalent to the following relation between the structure constants of $\mathfrak{g}$ and $\mathfrak{g}^*$ (the formula below is for a purely even $\mathfrak{g}$):

$$c^i_{jk} b^{nm}_i - c^n_{ji} b^{im}_k + c^m_{ji} b^{in}_k + c^n_{ki} b^{im}_j - c^m_{ki} b^{in}_j = 0. \tag{2.2}$$

THEOREM 2.1 (Drinfeld). *Let $\mathfrak{g}$ be a Lie bialgebra. Then on $\mathfrak{g} \oplus \mathfrak{g}^*$ there is a unique structure of a Lie bialgebra described by the properties:*

1. *$\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie subalgebras,*
2. *The natural inner product in $\mathfrak{g} \oplus \mathfrak{g}^*$ is* ad*-invariant,*
3. *The cobracket δ for $\mathfrak{g} \oplus \mathfrak{g}^*$ is given by the coboundary of the element $r = e_i \wedge e^i \in \wedge^2(\mathfrak{g} \oplus \mathfrak{g}^*)$.*

A partial converse is given by the following theorem.

THEOREM 2.2 (Manin). *Let $\mathfrak{a}, \mathfrak{b}$ be (finite-dimensional) Lie algebras such that the vector space $\mathfrak{d} := \mathfrak{a} \oplus \mathfrak{b}$ has a structure of a Lie algebra with an invariant inner product. Suppose that*

1. *$\mathfrak{a}$ and $\mathfrak{b}$ are isotropic subspaces in $\mathfrak{d}$,*
2. *$\mathfrak{a}$ and $\mathfrak{b}$ as Lie algebras are subalgebras in $\mathfrak{d}$.*

Then $\mathfrak{b} \cong \mathfrak{a}^*$, and $\mathfrak{a}$, $\mathfrak{b} \cong \mathfrak{a}^*$ are dual Lie bialgebras. The Lie algebra structure on $\mathfrak{d}$ coincides with the one given by Drinfeld's theorem.

The space $\mathfrak{d} = \mathfrak{d}(\mathfrak{g}) := \mathfrak{g} \oplus \mathfrak{g}^*$ with the bialgebra structure is called the (classical) *Drinfeld double* of the bialgebra $\mathfrak{g}$; the triple of Lie algebras $(\mathfrak{a}, \mathfrak{b}, \mathfrak{d} = \mathfrak{a} \oplus \mathfrak{b})$ is called

a *Manin triple.* Explicit formulas for the bracket in $\mathfrak{d}$:

$$[e_i, e_j] = c_{ij}^k e_k, \tag{2.3}$$

$$[e^i, e^j] = b_k^{ij} e^k, \tag{2.4}$$

$$[e_i, e^j] = b_i^{jk} e_k - c_{ik}^j e^k. \tag{2.5}$$

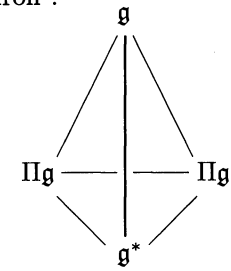
The cobracket $\delta\colon \mathfrak{d} \to \wedge^2\mathfrak{d}$ is given by

$$\delta(e_i) = b_i^{jk} e_j \wedge e_k \tag{2.6}$$

$$\delta(e^i) = -c_{jk}^i e^j \wedge e^k. \tag{2.7}$$

(Due to the natural inner product, the cobracket (2.6,2.7) gives rise to the second bracket in $\mathfrak{d}$: $[e_i, e_j]_{(2)} = -c_{ij}^k e_k$, $[e^i, e^j]_{(2)} = b_k^{ij} e^k$, $[e_i, e^j]_{(2)} = 0$.)

Now, to get a different formulation suitable for further generalizations, we first have to describe a Lie algebra structure geometrically. To this end, let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a vector space (from now on we return to the general super case). Consider it together with its neighbors. (The term "neighbor" for super vector spaces is due to Manin.) We have a tetrahedron[3]:



THEOREM 2.3. *A Lie (super)algebra structure in the vector space* $\mathfrak{g}$ *is equivalent to either of:*

(1) *A linear Poisson bracket on the manifold* $\mathfrak{g}^*$ *(Lie coalgebra).*
(2) *A linear Schouten bracket on the manifold* $\Pi\mathfrak{g}^*$ *(Lie anticoalgebra).*
(3) *A quadratic homological vector field on the manifold* $\Pi\mathfrak{g}$ *(Lie antialgebra).*

Recall that a *homological vector field* on a supermanifold M is an odd vector field (=odd derivation of the algebra of functions) $\hat{Q}$ such that $[\hat{Q}, \hat{Q}] = 2\hat{Q}^2 = 0$. A basis e_i in $\mathfrak{g}$ gives rise to linear coordinates on the supermanifolds $\Pi\mathfrak{g}$, $\mathfrak{g}^*$, and $\Pi\mathfrak{g}^*$, which we denote ξ^i, x_i and ξ_i, respectively. If $\tilde{e}_i =: \tilde{\imath}$, then $\tilde{\xi}^i = \tilde{\imath} + 1$, $\tilde{x}_i = \tilde{\imath}$, $\tilde{\xi}_i = \tilde{\imath} + 1$.

EXAMPLE 2.1. The homological field corresponding to a Lie superalgebra $\mathfrak{g}$ with the structure constants c_{ij}^k (i.e., $[e_i, e_j] = c_{ij}^k e_k$) is

$$\hat{Q} = \frac{1}{2}(-1)^{\tilde{\jmath}} \xi^j \xi^i c_{ij}^k \frac{\partial}{\partial \xi^k}. \tag{2.8}$$

The condition $\hat{Q}^2 = 0$ is equivalent to the Jacobi identity for c_{ij}^k. An element $X \in \mathfrak{g}$ (here we treat $\mathfrak{g}$ as a vector space, not as a manifold) corresponds to a vector field on $\Pi\mathfrak{g}$ (of parity opposite to that of X) with constant coefficients:

[3]The picture of a tetrahedron was suggested to me by Alan Weinstein instead of the three-pointed star with center at $\mathfrak{g}$ that I used initially.

$X = X^i e_i \mapsto i_X := (-1)^{\tilde{X}} X^i \partial/\partial\xi^i$. The odd linear map $i\colon \mathfrak{g} \to \mathrm{Vect}(\Pi\mathfrak{g})$ is, obviously, injective. The Lie bracket in $\mathfrak{g}$ can be reconstructed from the commutator of vector fields on $\Pi\mathfrak{g}$ by the "derived bracket" formula:

$$i_{[X,Y]} = [i_X, [\hat{Q}, i_Y]] = [[i_X, \hat{Q}], i_Y]. \tag{2.9}$$

Here at the l.h.s. [,] stands for the Lie bracket in $\mathfrak{g}$ with the structure constants c_{ij}^k, and at the r.h.s. [,] stands for the canonical Lie bracket (commutator) of vector fields. (Notice that vector fields with constant coefficients have zero commutator.) One can recognize in (2.9) an analog of Cartan's formulas, see Remark 1.2.

EXAMPLE 2.2. For the general linear algebra $\mathfrak{gl}(n)$ we have the field

$$\hat{Q} = -\sum \xi^{ik}\xi^{kj}\frac{\partial}{\partial\xi^{ij}}. \tag{2.10}$$

It is tangent to linear submanifolds that correspond to Lie subalgebras in $\mathfrak{gl}(n)$. Same formula works in the super case.

In a more conventional language, the vector field $\hat{Q}$ on $\Pi\mathfrak{g}$ coincides with the Chevalley–Eilenberg differential in $C^*(\mathfrak{g}) = C^*(\mathfrak{g}; \mathbb{R})$, for ordinary Lie algebras. For Lie superalgebras this can be taken as a convenient definition.

EXAMPLE 2.3. Consider a general odd (formal) vector field on $\mathbb{R}^{n|m}$:

$$\hat{Q} = \left(Q_0^k + \xi^i Q_i^k + \frac{1}{2}\xi^j\xi^i Q_{ij}^k + \frac{1}{3!}\xi^l\xi^j\xi^i Q_{ijl}^k + \dots\right)\frac{\partial}{\partial\xi^k}. \tag{2.11}$$

The coefficients Q_0^k, Q_i^k, Q_{ij}^k, Q_{ijl}^k, ... define a sequence of N-ary operations ($N = 0, 1, 2, 3, \dots$) on the vector space $\mathbb{R}^{m|n}$, and the condition $\hat{Q}^2 = 0$ expands to a linked sequence of "Jacobi identities". If only the quadratic term in the Maclaurin expansion (2.11) is present, we return to the case of a Lie algebra. The general case is a *strong homotopy Lie algebra* (*L_∞-algebra*) (due to Stasheff, see [**13**]).

EXAMPLE 2.4. If instead of commutative algebras of functions on $\mathbb{R}^{n|m}$ one considers free noncommutative associative algebras and their odd derivations of square zero (analog of homological fields), then a homotopy version of associative algebras (so called *A_∞-algebras*) will be obtained [**24**]. Again, the purely quadratic case reproduces ordinary (associative) algebras.

Now we can say what is, in this language, a general "bi-" structure for a Lie algebra. Consider the above tetrahedron of vector spaces. A "bi-" structure for $\mathfrak{g}$ is exactly a Lie algebra structure for two of its vertices, with a certain compatibility condition. In view of Theorem 2.3, up to renaming, the list of the corresponding geometric structures is exhausted by three cases:

I.	$\mathfrak{g}$, $\mathfrak{g}^*$ Lie algebras	On $\Pi\mathfrak{g}$ (or $\Pi\mathfrak{g}^*$) there are homological field $\hat{Q}$ and odd bracket.	QS-manifold
II.	$\mathfrak{g}$, $\Pi\mathfrak{g}^*$ Lie algebras	On $\mathfrak{g}^*$ (or $\Pi\mathfrak{g}$) there are homological field $\hat{Q}$ and even bracket.	QP-manifold
III.	$\mathfrak{g}$, $\Pi\mathfrak{g}$ Lie algebras	On $\mathfrak{g}^*$ (or $\Pi\mathfrak{g}^*$) there are even and odd brackets.	PS-manifold

One should explain the compatibility conditions. For a pair consisting of a Schouten or Poisson bracket and a homological vector field there is only one reasonable condition, that the field $\hat{Q}$ should be a derivation of the bracket. We call Poisson (Schouten) manifolds endowed with a homological field which is a derivation of the bracket *QP-manifolds* (*QS-manifolds,* respectively).

THEOREM 2.4. *A QS-manifold structure on $\Pi\mathfrak{g}$ with a linear Schouten bracket and a quadratic field $\hat{Q}$ is equivalent to a Lie bialgebra structure for $\mathfrak{g}$.*

PROOF. As noted above, the vector field $\hat{Q}$ corresponds to the Chevalley–Eilenberg differential for the trivial representation. The Lie derivative $L_{\hat{Q}}$ corresponds to the Chevalley–Eilenberg differential for other (nontrivial) representations. Thus the condition that the cobracket on $\mathfrak{g}$ is a cocycle is equivalent to $L_{\hat{Q}}S = 0$, for the Hamiltonian S that defines the corresponding Lie–Schouten structure on $\Pi\mathfrak{g}$. On the other hand, we have, by a straightforward calculation,

$$\hat{Q}\{f,g\}_S = \{\hat{Q}f,g\}_S + (-1)^{\tilde{f}+1}\{f,\hat{Q}g\}_S + (-1)^{\tilde{f}}\{f,\{L_{\hat{Q}}S,g\}\}, \tag{2.12}$$

for the bracket $\{\ ,\ \}_S$ defined by S. Hence, if $L_{\hat{Q}}S = 0$, then the derivation property holds. Conversely, if the derivation property holds, then for all f, g $\{f,\{L_{\hat{Q}}S,g\}\} = 0$, and this implies $L_{\hat{Q}}S = 0$ by the virtue of the non-degeneracy of the canonical Poisson bracket. □

In particular, this is the way to get the correct formulas like (2.2) in the super case $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Practically, this theorem can be used instead of the definition in this case.

What about the QP-manifolds? They correspond to an "odd version" of the Lie bialgebra notion.

DEFINITION 2.2. An *odd Lie bialgebra* is a Lie superalgebra $\mathfrak{g}$ such that the space $\Pi\mathfrak{g}^*$ is also a Lie superalgebra so that on the antialgebra $\Pi\mathfrak{g}$ is induced a QP-structure (with linear Poisson bracket and quadratic homological field).

We shall elaborate this in Section 6.

REMARK 2.1. It should be emphasized that while a Lie algebra $\mathfrak{g}$ can be viewed via three geometric manifestations, listed in Theorem 2.3, for a "bi-" case there is just one geometric picture. A QS-structure with linear bracket and quadratic field (realized, equivalently, either on $\Pi\mathfrak{g}$ or on $\Pi\mathfrak{g}^*$) corresponds to a Drinfeld's Lie bialgebra, and a QP-structure with linear bracket and quadratic field (realized, equivalently, either on $\Pi\mathfrak{g}$ or on $\mathfrak{g}^*$) corresponds to an odd bialgebra.

We reserve the name *PS-manifold* for a manifold with a pair of brackets of opposite parity. The precise compatibility condition in this case is not obvious. We refrain from discussing it here. Geometry of spaces with even and odd bracket was studied, in particular, in [7]. For $\mathfrak{g} \oplus \Pi\mathfrak{g}$, an odd endomorphism replaces the pairing.

Now, we need the description of Drinfeld's double in this language. Start from the space $\Pi\mathfrak{g}$. Notice that $\Pi(\mathfrak{g} \oplus \mathfrak{g}^*) \cong T^*\Pi\mathfrak{g}$. The natural (symmetric) inner product on $\mathfrak{g} \oplus \mathfrak{g}^*$ corresponds to the canonical symplectic structure on $T^*\Pi\mathfrak{g}$. Denote coordinates on $\Pi\mathfrak{g}$ by ξ^i and their conjugate momenta by ξ_j. As we know from Section 1, a linear Schouten bracket on $\Pi\mathfrak{g}$ is specified by an odd Hamiltonian of the form $S = (1/2)\xi^l S_l^{nm}\xi_m\xi_n \in C^\infty(T^*\Pi\mathfrak{g})$. On the other hand, a vector field $\hat{Q} = (1/2)\xi^j\xi^i Q_{ij}^k \partial/\partial\xi^k \in \mathrm{Vect}(\Pi\mathfrak{g})$ induces the Hamiltonian $Q = (1/2)\xi^j\xi^i Q_{ij}^k\xi_k \in C^\infty(T^*\Pi\mathfrak{g})$.

THEOREM 2.5. *The following statements hold for* $T^*\Pi\mathfrak{g} \cong \Pi(\mathfrak{g} \oplus \mathfrak{g}^*)$:

1. *The QS-condition for* $\Pi\mathfrak{g}$ *with the field* $\hat{Q}$ *and the Schouten tensor* S *is equivalent to the vanishing of the Poisson bracket*

$$\{Q, S\} = \frac{1}{4}\{\xi^j \xi^i Q_{ij}^k \xi_k, \xi^l S_l^{nm} \xi_m \xi_n\} = 0. \tag{2.13}$$

2. *The function* $= Q + S$ *on* $T^*\Pi\mathfrak{g}$ *satisfies* $\{Q+S, Q+S\} = 0$, *so the Hamiltonian vector field* $\hat{Q}_D := X_{Q+S} \in \mathrm{Vect}(T^*\Pi\mathfrak{g})$ *is homological. It corresponds to the Lie algebra structure in Drinfeld's double* $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$.

PROOF. To prove part 1, notice that $\{Q, S\} = L_{\hat{Q}} S$, thus the equation $\{Q, S\} = 0$ is equivalent to the QS-condition (see the proof of Theorem 2.4). To prove part 2, notice that the equation $\{Q+S, Q+S\} = 0$ implies the equation $\hat{Q}_D^2 = X_{Q+S}^2 = 0$ for the corresponding Hamiltonian field. Finally, to show that $\hat{Q}_D$ gives exactly the Lie bracket in Drinfeld's double, we compare the explicit formulas. For X_Q we obtain, using the general formula (1.16):

$$\begin{aligned} X_Q = \{Q, \ \} &= (-1)^{(\tilde{k}+1)(\tilde{Q}+1)} \frac{\partial Q}{\partial \xi_k} \frac{\partial}{\partial \xi^k} - (-1)^{(\tilde{k}+1)\tilde{Q}} \frac{\partial Q}{\partial \xi^k} \frac{\partial}{\partial \xi_k} \\ &= \frac{\partial Q}{\partial \xi_k} \frac{\partial}{\partial \xi^k} + (-1)^{\tilde{k}} \frac{\partial Q}{\partial \xi^k} \frac{\partial}{\partial \xi_k} = \frac{1}{2} \xi^j \xi^i Q_{ij}^k \frac{\partial}{\partial \xi^k} + (-1)^{\tilde{k}} \xi^i Q_{ik}^j \xi_j \frac{\partial}{\partial \xi_k} \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} X_S = \{S, \ \} &= \frac{\partial S}{\partial \xi_k} \frac{\partial}{\partial \xi^k} + (-1)^{\tilde{k}} \frac{\partial S}{\partial \xi^k} \frac{\partial}{\partial \xi_k} \\ &= \xi_n \xi^l S_l^{nk} \frac{\partial}{\partial \xi^k} + (-1)^{\tilde{k}} \frac{1}{2} S_k^{nm} \xi_m \xi_n \frac{\partial}{\partial \xi_k} \end{aligned}$$

Thus $\hat{Q}_D = X_Q + X_S$ reads

$$\hat{Q}_D = \left(\frac{1}{2} \xi^j \xi^i Q_{ij}^k + \xi_j \xi^i S_i^{jk} \right) \frac{\partial}{\partial \xi^k} + (-1)^{\tilde{k}} \left(\frac{1}{2} S_k^{ij} \xi_j \xi_i + \xi^i Q_{ik}^j \xi_j \right) \frac{\partial}{\partial \xi_k}, \tag{2.14}$$

and in the case of a purely even $\mathfrak{g}$ it is reduced to

$$\left(\frac{1}{2} \xi^j \xi^i Q_{ij}^k + \xi_j \xi^i S_i^{jk} \right) \frac{\partial}{\partial \xi^k} + \left(\frac{1}{2} S_k^{ij} \xi_j \xi_i - \xi_j \xi^i Q_{ik}^j \right) \frac{\partial}{\partial \xi_k},$$

which reproduces formulas (2.3–2.5) for the Lie bracket. □

REMARK 2.2. It should be noted that the description of the double of a Lie bialgebra in terms of a pair of commuting Hamiltonians was first done in [**14**]. A similar description for Lie "quasi-bialgebras" was obtained in [**9**].

Theorem 2.5 gives for $T^*\Pi\mathfrak{g}$ a Q-structure corresponding to the Lie bracket in the double. To get on $T^*\Pi\mathfrak{g}$ an S-structure corresponding to the cobracket, consider the cotangent bundle of $T^*\Pi\mathfrak{g}$. Denote by π_i, π^i the conjugate momenta for ξ^i, ξ_i, respectively. Notice that π_i transforms as ξ_i, and π^i as $(-1)^{\tilde{i}}\xi^i$ (see a more general statement in Section 3). Hence $r = -(-1)^{\tilde{i}}\pi^i \pi_i = \pi_i \pi^i$ is an invariantly defined function on $T^*T^*\Pi\mathfrak{g}$. Consider the following linear map $C^\infty(T^*\Pi\mathfrak{g}) \to C^\infty(T^*T^*\Pi\mathfrak{g})$, $f \mapsto \bar{f} := \{p_{X_f}, r\}$.

LEMMA 2.1. *For an arbitrary function $f \in C^\infty(T^*\Pi\mathfrak{g})$ there is an explicit formula*

$$\bar{f} = \{p_{X_f}, r\} = (-1)^{(\tilde{\imath}+\tilde{\jmath})\tilde{f}} \frac{\partial^2 f}{\partial\xi^j \partial\xi^i}\pi^i\pi^j - (-1)^{(\tilde{\imath}+\tilde{\jmath})(\tilde{f}+1)} \frac{\partial^2 f}{\partial\xi_j \partial\xi_i}\pi_i\pi_j. \tag{2.15}$$

PROOF. Notice, first, that the bracket of an arbitrary $g \in C^\infty(T^*T^*\Pi\mathfrak{g})$ with $r = \pi_i\pi^i$ equals

$$\{g, r\} = -(-1)^{(\tilde{\jmath}+1)\tilde{g}} \frac{\partial g}{\partial\xi^j}\pi^j - (-1)^{(\tilde{\jmath}+1)(\tilde{g}+1)} \frac{\partial g}{\partial\xi_j}\pi_j. \tag{2.16}$$

Now, for an arbitrary $f \in C^\infty(T^*\Pi\mathfrak{g})$ we get the function

$$p_{X_f} = (-1)^{(\tilde{\imath}+1)(\tilde{f}+1)} \frac{\partial f}{\partial\xi_i}\pi_i - (-1)^{(\tilde{\imath}+1)\tilde{f}} \frac{\partial f}{\partial\xi^i}\pi^i \tag{2.17}$$

on $T^*T^*\Pi\mathfrak{g}$. Substituting (2.17) into (2.16), we obtain formula (2.15) by a direct simplification (which we omit). Notice that the terms containing "cross" products like $\pi_i\pi^j$ remarkably cancel. □

THEOREM 2.6. *Consider $S_D := (1/2)\{Q_D, r\} \in C^\infty(T^*T^*\Pi\mathfrak{g})$, where $Q_D \in C^\infty(T^*T^*\Pi\mathfrak{g})$ corresponds to the vector field $\hat{Q}_D$. Then S_D is a Schouten tensor for $T^*\Pi\mathfrak{g}$, and together with the field $\hat{Q}_D$ it gives a QS-structure on $T^*\Pi\mathfrak{g}$. The Schouten bracket specified by S_D corresponds to the cobracket in Drinfeld's double $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$.*

PROOF. We have $S_D = (1/2)(\overline{Q+S}) = (1/2)(\bar{Q} + \bar{S})$, in the notation of Lemma 2.1. Applying the lemma to the functions Q and S, we obtain the formulas

$$\bar{Q} = (-1)^{\tilde{\imath}+\tilde{\jmath}} Q^k_{ij}\xi_k\pi^j\pi^i = (-1)^{\tilde{\jmath}}\pi^j(-1)^{\tilde{\imath}}\pi^i Q^k_{ij}\xi_k \tag{2.18}$$

$$\bar{S} = -\xi^k S^{ij}_k \pi_j\pi_i. \tag{2.19}$$

Here in (2.18) we changed the order and separated the factors $(-1)^{\tilde{\imath}}\pi^i$. The point is that the correspondence $\xi^i \leftrightarrow (-1)^{\tilde{\imath}}\pi^i$ respects the Poisson bracket: $\{\xi_i, \xi^j\}_{T^*\Pi\mathfrak{g}} = \delta^j_i = \{\xi_i, (-1)^{\tilde{\jmath}}\pi^j\}_{T^*T^*\Pi\mathfrak{g}}$. In the same way, $\{\xi_i, \xi^j\}_{T^*\Pi\mathfrak{g}} = \delta^j_i = \{\pi_i, \xi^j\}_{T^*T^*\Pi\mathfrak{g}}$. Hence $\{\bar{Q}, \bar{Q}\} = 4\{Q, Q\} = 0$, $\{\bar{S}, \bar{S}\} = 4\{S, S\} = 0$ by the virtue of these correspondences, and $\{\bar{Q}, \bar{S}\} = 0$ because $\bar{Q}$ and $\bar{S}$ do not contain conjugate variables. Thus, $\{S_D, S_D\} = 0$. The identity $\{Q_D, S_D\} = 0$ holds by the construction of S_D. To notice that S_D corresponds to the cobracket in Drinfeld's double (up to a common sign), it suffices to compare the explicit formula obtained

$$S_D = \frac{1}{2}(-1)^{\tilde{\imath}+\tilde{\jmath}} Q^k_{ij}\xi_k\pi^j\pi^i - \frac{1}{2}\xi^k S^{ij}_k \pi_j\pi_i \tag{2.20}$$

with formulas (2.6,2.7) in the even case. □

(It is clear now how to get an "odd analog" of Drinfeld's double. Let $\mathfrak{g}$ be an odd bialgebra, so $\Pi\mathfrak{g}$ is a QP-manifold with the quadratic $\hat{Q}$ and linear bracket. Instead of $T^*\Pi\mathfrak{g}$ we consider the supermanifold $\Pi T^*\Pi\mathfrak{g} \cong \Pi\mathfrak{g} \oplus \mathfrak{g}^*$. In a way similar to the above, it can be proved that $\Pi T^*\Pi\mathfrak{g}$ has a desired QP-structure, corresponding to an odd bialgebra structure in $\mathfrak{g} \oplus \Pi\mathfrak{g}^*$. The space $\mathfrak{g} \oplus \Pi\mathfrak{g}^*$ with this structure will be called the *odd double* of an odd bialgebra $\mathfrak{g}$. We will return to this construction in Section 6 and give details.)

REMARK 2.3. Drinfeld's classical double is, by the construction, a *coboundary Lie bialgebra* (see [**3**]). That means that $S_D = \{Q_D, r/2\}$, in our language. Suppose that for an arbitrary Lie algebra $\mathfrak{g}$ an even Hamiltonian $r \in C^\infty(T^*\Pi\mathfrak{g})$ of the form $r = r^{ij}\pi_i\pi_j$ (with constant r^{ij}) is considered. The Poisson bracket with the odd Hamiltonian $Q = (1/2)\xi^j\xi^i Q_{ij}^k\pi_k$ corresponding to the Lie algebra structure, $\{Q, r/2\} =: S$, is quadratic in π_i: $S = (1/2)\xi^i S_i^{jk}\pi_k\pi_j$. The linear odd bracket on $\Pi\mathfrak{g}$ defined as $\{f, g\}_S = \{f, \{S, g\}\}$ is automatically compatible with Q. It satisfies the Jacobi identity (thus making $\mathfrak{g}$ into a Lie bialgebra) if and only if the Poisson bracket $\{\{Q, r\}, \{Q, r\}\}$ vanishes. Since $\{Q, \{r, \{Q, r\}\}\} = \{\{Q, r\}, \{Q, r\}\}$, this is equivalent to

$$\{Q, \{r, r\}_Q\} = 0 \tag{2.21}$$

where the inner bracket denotes the operation

$$r_1, r_2 \mapsto \{r_1, r_2\}_Q := \{r_1, \{Q, r_2\}\} = \{\{r_1, Q\}, r_2\} \tag{2.22}$$

on Hamiltonians r_1, r_2 depending only on π_i (this is exactly the Lie–Schouten bracket on $\Pi\mathfrak{g}^*$). The equation (2.21) is satisfied if, in particular,

$$\{r, r\}_Q = 0. \tag{2.23}$$

This is the *classical Yang–Baxter equation* for r, while (2.21) is the "generalized" classical Yang–Baxter equation. Notice that $r = \pi_i\pi^i$ for Drinfeld's double satisfies (2.21), but it does not satisfy (2.23), $\{r, r\}_{Q_D}$ being a certain canonical $\hat{Q}_D$-invariant cubic expression in π_i, π^j (in the standard language r is said to be a solution of the "modified" classical Yang–Baxter equation).

3. Doubles for Lie bialgebroids

In this section we review the results of D. Roytenberg [**22**] and A. Vaintrob [**26**], who independently applied super methods to Lie bialgebroids. As noted before, we give a "superized" version, i.e., consider from the beginning super Lie (bi)algebroids over supermanifolds. Lie bialgebroids were originally introduced by Mackenzie and Xu [**17**]. Important results, which particularly nicely fit into the super framework (but were originally formulated without it), are due to Y. Kosmann-Schwarzbach [**10**].

Consider a Lie algebroid $E \to M$ as defined in Section 1. In direct analogy with the tangent bundles and Lie algebras, it gives rise to a differential algebra, which is $C^\infty(\Pi E)$ with the homological vector field $\hat{Q} = \hat{Q}_E \in \mathrm{Vect}(\Pi E)$,

$$\hat{Q} = \xi^i Q_i^a(x)\frac{\partial}{\partial x^a} + \frac{1}{2}\xi^j\xi^i Q_{ij}^k(x)\frac{\partial}{\partial \xi^k}. \tag{3.1}$$

(Here we use the notation x^a for coordinates in the base and ξ^i for coordinates in the fiber of ΠE.) In the purely even case, $C^\infty(\Pi E)$ is the algebra of smooth sections of the exterior bundle $C^\infty(M, \Lambda(E^*))$. The differential $d_E = \hat{Q}_E\colon C^\infty(M, \Lambda(E^*)) \to C^\infty(M, \Lambda(E^*))$ is analogous to the de Rham and Chevalley–Eilenberg differentials (which are both due to Cartan). The coefficients $Q_{ij}^k(x)$ describe the Lie brackets of a local basis of sections in E, while $Q_i^a(x)$ correspond to the anchor map $a\colon C^\infty(M, E) \to \mathrm{Vect}(M)$. The bracket $[\ ,\ ]_E$ and the anchor are recovered from

$\hat{Q}$ by the formulas similar to (2.9):

$$a(u)f = (-1)^{\tilde{u}}[i_u, [\hat{Q}, f]] = (-1)^{\tilde{u}}[[i_u, \hat{Q}], f] \tag{3.2}$$

$$i_{[u,v]_E} = [i_u, [\hat{Q}, i_v]] = [[i_u, \hat{Q}], i_v] \tag{3.3}$$

where $i_u = (-1)^{\tilde{u}} u^i(x)\partial/\partial\xi^i$ for $u = u^i(x)e_i \in C^\infty(M, E)$. In particular, for the basis sections we have

$$a(e_i) = Q_i^a(x)\frac{\partial}{\partial x^a}, \quad [e_i, e_j]_E = (-1)^{\tilde{j}} Q_{ij}^k(x) e_k. \tag{3.4}$$

The identity $\hat{Q}^2 = 0$ compactly contains all the properties of the anchor and the bracket. Thus, we can take it as an equivalent definition of a Lie algebroid structure. A vector bundle endowed with $\hat{Q}$ of the form (3.1) such that $\hat{Q}^2 = 0$ is, by definition, a *Lie antialgebroid*. $\Pi E \to M$ is the Lie antialgebroid corresponding to the Lie algebroid $E \to M$.

The Lie algebroid structure in E also defines a Schouten bracket in the algebra $C^\infty(\Pi E^*)$, which in the purely even case is the algebra of smooth sections $C^\infty(M, \Lambda(E))$, and a Poisson bracket in the algebra $C^\infty(E^*)$. Thus, as for Lie algebras, there are a Q-manifold, an S-manifold and a P-manifold associated with a Lie algebroid.

Return for a moment to a purely even situation. Let the dual bundle $E^* \to M$ also have a structure of a Lie algebroid. Then, there is a differential

$$d_{E^*}\colon C^\infty(M, \Lambda(E)) \to C^\infty(M, \Lambda(E)).$$

It can be applied, in particular, to sections of $E \to M$ considered as sections of the exterior bundle.

DEFINITION 3.1 (Mackenzie–Xu [**17**]). A *Lie bialgebroid* over a manifold M is a Lie algebroid $E \to M$ for which the dual bundle $E^* \to M$ is also a Lie algebroid so that the differential d_{E^*} satisfies the derivation property for the bracket of sections of E.

It was shown by Y. Kosmann-Schwarzbach [**10**] that it is equivalent and more convenient to require that d_{E^*} is a derivation of the Schouten bracket in the whole algebra $C^\infty(M, \Lambda(E))$. It can also be shown (though it is not straightforward at this point) that E and E^* here can be interchanged. Assuming this, the definition can be put into a super language (allowing to cover the general case) as follows:

DEFINITION 3.2 (equivalent). A (super) *Lie bialgebroid* over a supermanifold M is a vector bundle $E \to M$ such that the space of the opposite vector bundle $\Pi E \to M$ is a QS-manifold, with the homological field of the form

$$\hat{Q} = \xi^i Q_i^a(x)\frac{\partial}{\partial x^a} + \frac{1}{2}\xi^j\xi^i Q_{ij}^k(x)\frac{\partial}{\partial \xi^k} \tag{3.5}$$

and the non-vanishing Schouten brackets of coordinates of the form

$$\{\xi^i, x^a\}_S = (-1)^{\tilde{i}} S^{ia}(x), \quad \{\xi^i, \xi^j\}_S = (-1)^{\tilde{i}} \xi^k S_k^{ij}(x). \tag{3.6}$$

That means that the bracket is given by the Hamiltonian $S = S_E \in C^\infty(T^*\Pi E)$,

$$S = S^{ia}(x) p_a \pi_i + \frac{1}{2}\xi^k S_k^{ij}(x)\pi_j\pi_i. \tag{3.7}$$

THEOREM 3.1. *E is a Lie bialgebroid if and only if the Poisson bracket $\{Q, S\}$ vanishes, where $Q = p(\hat{Q}) \in C^\infty(T^*\Pi E)$ is the Hamiltonian corresponding to the vector field $\hat{Q}$.*

PROOF. Similarly to Theorem 2.4: the condition of a Lie bialgebroid is equivalent to $L_{\hat{Q}_E} S_E = 0$, and $L_{\hat{Q}_E} S_E = \{Q, S\}$. □

This definition is not manifestly symmetric with respect to E and E^*. A symmetric description can be achieved with the help of the following theorem (see Mackenzie and Xu [**17**]; we provide a proof in the Appendix).

THEOREM 3.2. *For an arbitrary vector bundle $E \to M$ there is a natural diffeomorphism of the cotangent bundles $F\colon T^*E \cong T^*E^*$, which preserves the symplectic structure. In coordinates:*

$$F\colon (x^a, y^i, p_a, p_i) \mapsto (x^a, y_i, p_a, p^i) = (x^a, p_i, p_a, -(-1)^{\tilde{\imath}} y^i).$$

Here y^i stand for (left) coordinates in the fiber of E, y_i stand for the contragredient right coordinates in the fiber of E^*, and p_i, p^i for the respective conjugate momenta.

Consider a Lie algebroid ΠE. Let the vector field $\hat{Q} = \hat{Q}_E \in \mathrm{Vect}(\Pi E)$ be given by (3.5). The corresponding Hamiltonian $Q_E \in C^\infty(T^*\Pi E)$ is

$$Q_E = \xi^i Q_i^a(x)\, p_a + \frac{1}{2}\xi^j \xi^i Q_{ij}^k(x)\, \pi_k. \tag{3.8}$$

We shall apply Theorem 3.2 to the vector bundle ΠE. Suppose E^* is also a Lie algebroid. Let $\hat{Q}_{E^*} \in \mathrm{Vect}(\Pi E^*)$ be the homological field defining the algebroid structure in E^* and the corresponding Hamiltonian $Q_{E^*} \in C^\infty(T^*\Pi E^*)$ be

$$Q_{E^*} = \xi_i Q^{ia}(x) p_a + \frac{1}{2}\xi_j \xi_i Q_k^{ij}(x)\pi^k. \tag{3.9}$$

Here ξ_i with lower indices denote the coordinates in the fiber of ΠE^* and π^i the conjugate momenta. In particular, the anchor and bracket in E^* will be given by

$$a(e^i) = Q^{ia}(x)\frac{\partial}{\partial x^a}, \quad [e^i, e^j]_{E^*} = (-1)^{\tilde{\jmath}} Q_k^{ij}(x) e^k. \tag{3.10}$$

By Theorem 3.2, we identify $T^*\Pi E$ and $T^*\Pi E^*$. Rewritten in coordinates $\xi^i = (-1)^{\tilde{\imath}}\pi^i, \pi_i = \xi_i$, the Hamiltonian Q_{E^*} becomes

$$Q_{E^*} = \pi_i Q^{ia}(x) p_a + \frac{1}{2}\pi_j \pi_i Q_k^{ij}(x)(-1)^{\tilde{k}}\xi^k = \tag{3.11}$$

$$Q^{ia}(x) p_a \pi_i + \frac{1}{2}(-1)^{\tilde{k}} Q_k^{ij}(x)\xi^k \pi_j \pi_i.$$

THEOREM 3.3 (Roytenberg [**22**], Vaintrob [**26**]). *The Hamiltonians S_E and Q_{E^*} coincide. Hence, E is a Lie bialgebroid if and only if the Hamiltonians $Q_E, Q_{E^*} \in C^\infty(T^*\Pi E) = C^\infty(T^*\Pi E^*)$ commute.*

PROOF. The key statement is that $Q_{E^*} = S_E$ (after the identification of $T^*\Pi E$ and $T^*\Pi E^*$). The second statement would follow from Theorem 3.1. We have to show that the Schouten bracket on ΠE induced by the algebroid structure in E^* and specified by S_E is the same as given by the quadratic Hamiltonian (3.11). To this end, we check the non-vanishing Schouten brackets for coordinates. We get $\{\xi^i, x^a\}_S = \{\theta(e^i), x^a\}_S = (-1)^{\tilde{\imath}} a(e^i) x^a = (-1)^{\tilde{\imath}} Q^{ia}$ and $\{\xi^i, \xi^j\}_S = \theta([e^i, e^j]_{E^*}) = \theta((-1)^{\tilde{\jmath}} Q_k^{ij} e^k) = (-1)^{\tilde{\imath}+\tilde{k}} Q_k^{ij}\xi^k$, by the definition of the Schouten bracket induced

by an algebroid structure and formulas (3.10). (Here θ: $C^\infty(M, E^*) \to C^\infty(\Pi E)$ is $\alpha = e^i\alpha_i(x) \mapsto \theta(\alpha) = \xi^i\alpha_i(x)$.) This exactly coincides with the Schouten brackets given by the Hamiltonian (3.11). Hence, $S_E = Q_{E^*}$. □

It follows that the condition of a Lie bialgebroid is self-dual. Under the identification $T^*\Pi E = T^*\Pi E^*$ the QS-structure on ΠE corresponds to the QS-structure on ΠE^* so that the homological field and the bracket effectively exchange places. Note, however, that this self-duality is an extra statement not necessary for writing the condition of a Lie bialgebroid in the Poisson form $\{Q, S\} = 0$. It does not make sense in a nonlinear case considered in Section 5.

How to get an analog of Drinfeld's double for a bialgebroid E? Roytenberg suggested the following argument. Consider the Hamiltonian vector fields on the supermanifold $T^*\Pi E$ corresponding to the Hamiltonians $Q = Q_E$ and $S = Q_{E^*}$. According to the general formulas, for an odd function on $T^*\Pi E$ we have

$$X_f = \{f, \ \} = \frac{\partial f}{\partial p_a}\frac{\partial}{\partial x^a} - (-1)^{\tilde{a}}\frac{\partial f}{\partial x^a}\frac{\partial}{\partial p_a} + \frac{\partial f}{\partial \pi_i}\frac{\partial}{\partial \xi^i} + (-1)^{\tilde{\imath}}\frac{\partial f}{\partial \xi^i}\frac{\partial}{\partial \pi_i}. \tag{3.12}$$

Hence, we get:

$$X_Q = \xi^i Q_i^a \frac{\partial}{\partial x^a} - (-1)^{\tilde{a}}\left(\frac{\partial Q_i^b}{\partial x^a}p_b\xi_i + \frac{1}{2}\frac{\partial Q_{ij}^k}{\partial x^a}\pi_k\xi^j\xi^i\right)\frac{\partial}{\partial p_a} + \frac{1}{2}\xi^j\xi^i Q_{ij}^k\frac{\partial}{\partial \xi^k} + (-1)^{\tilde{k}}\left(Q_k^a p_a + \xi^j Q_{jk}^i\pi_i\right)\frac{\partial}{\partial \pi_k}, \tag{3.13}$$

$$X_S = \pi_i Q^{ia}\frac{\partial}{\partial x^a} - (-1)^{\tilde{a}}\left(\frac{\partial Q^{ib}}{\partial x^a}p_b\pi_i + \frac{1}{2}(-1)^{\tilde{k}}\frac{\partial Q_k^{ij}}{\partial x^a}\xi^k\pi_j\pi_i\right)\frac{\partial}{\partial p_a} + \left(Q^{ka}p_a + (-1)^{\tilde{\imath}}\pi_j Q_i^{jk}\xi^i\right)\frac{\partial}{\partial \xi^k} + (-1)^{\tilde{k}}\frac{1}{2}\pi_j\pi_i Q_k^{ij}\frac{\partial}{\partial \pi_k}. \tag{3.14}$$

Mimicking the construction of Drinfeld's double for Lie bialgebras, define $\hat{Q}_D := X_Q + X_S$, and for the vector field $\hat{Q}_D$ on $T^*\Pi E$ we get

$$\begin{aligned}\hat{Q}_D = {} & \left(\xi^i Q_i^a + \pi_i Q^{ia}\right)\frac{\partial}{\partial x^a} \\ & - (-1)^{\tilde{a}}\left(\frac{\partial Q_i^b}{\partial x^a}p_b\xi_i + \frac{\partial Q^{ib}}{\partial x^a}p_b\pi_i + \frac{1}{2}\frac{\partial Q_{ij}^k}{\partial x^a}\pi_k\xi^j\xi^i + \frac{1}{2}(-1)^{\tilde{k}}\frac{\partial Q_k^{ij}}{\partial x^a}\xi^k\pi_j\pi_i\right)\frac{\partial}{\partial p_a} \\ & + \left(Q^{ka}p_a + (-1)^{\tilde{\imath}}\pi_j Q_i^{jk}\xi^i + \frac{1}{2}\xi^j\xi^i Q_{ij}^k\right)\frac{\partial}{\partial \xi^k} \\ & + (-1)^{\tilde{k}}\left(Q_k^a p_a + \xi^j Q_{jk}^i\pi_i + \frac{1}{2}\pi_j\pi_i Q_k^{ij}\right)\frac{\partial}{\partial \pi_k}.\end{aligned} \tag{3.15}$$

Obviously, the field $\hat{Q}_D$ is homological, because Q and S commute. Some terms in this lengthy expression are recognizable as similar to those in Drinfeld's double in the bialgebra case (compare with (2.14)). However, because of the presence of terms cubic in variables ξ^i, π_j, the homological field $\hat{Q}_D$ cannot be related with a Lie algebroid over M.

In Roytenberg's thesis [**22**] it was suggested to call the field $\hat{Q}_D \in \mathrm{Vect}(T^*\Pi E)$ the *Drinfeld double of the Lie bialgebroid E*.

It should be once again emphasized that the supermanifold $T^*\Pi E$ does not have the form ΠA for any vector bundle $A \to M$ and that the field $\hat{Q}_D$ contains terms not appropriate for Lie (anti)algebroids over M.

Two other constructions were suggested as analogs of Drinfeld's double. Liu, Weinstein and Xu [**15**] considered the direct sum $E \oplus E^*$ as a vector bundle over M and endowed it with a structure of a *Courant algebroid*, a notion that they defined. Roughly, it is a vector bundle with a bracket of sections which satisfies the Jacobi identity up to certain anomaly expressible in terms of a non-degenerate inner product (see [**15**], also [**22**]). There is a natural projection $T^*\Pi E \to \Pi(E \oplus E^*)$. Roytenberg [**22**] showed that the Courant algebroid structure in $E \oplus E^*$ is obtained by exactly the same procedure (described in Section 2 above) that reproduces the Lie bracket on the double of a Lie bialgebra in case the base manifold is a point. K. Mackenzie [**19, 20**] suggested to consider as a double of a Lie bialgebroid E the manifold T^*E together with the structure of a "double Lie algebroid", in a categorical sense precisely defined by him. The easy part of this notion as is follows. There is a diagram

$$\begin{array}{ccc} T^*E = T^*E^* & \longrightarrow & E^* \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array} \tag{3.16}$$

It can be seen that each arrow is a vector bundle projection, and the horizontal arrows constitute a morphism of the vertical bundle structure, and vice versa. Such structure is called a *double vector bundle*. Moreover, it turns out that there are Lie algebroid structures on all four sides, and, in addition, certain conditions are satisfied [**19, 20**] which allow to describe the diagram (3.16) as a *double Lie algebroid* over M (this is the hard part). Again, the "direct sum object" $E \oplus E^*$ sits inside this picture, due to the projection $T^*E \to E \oplus E^*$. (I hope to elucidate the relation of Mackenzie's beautiful picture with the super approach elsewhere.)

Since there is no hope to interpret the homological field $\hat{Q}_D$ in terms of a vector bundle over M, other approach should be taken. We think that the key hint is in the following fact. One can check that a change of coordinates in $T^*\Pi E$ (see the Appendix) preserves the following $\mathbb{Z}$-gradings: $\#p + \#\pi$ and $\#p + \#\xi$ (the symbol $\#$ means total degree in a given variable), as well as their linear combinations, viz., $\#\xi - \#\pi$ (cf. with the similar gradings considered in [**27, 28**].) It was observed in [**22**] that the vector field $\hat{Q}_D$ is homogeneous w.r.t. a grading defined as

$$(\#p + \#\pi) + (\#p + \#\xi) = 2\#p + \#\pi + \#\xi, \tag{3.17}$$

of degree $+1$. For a vector bundle $E \to M$, one can uniquely characterize the homological fields on ΠE of the form (3.1) corresponding to a Lie algebroid structures in E as having the degree $+1$ in linear fiber coordinates ξ^i.

Hence, *a $\mathbb{Z}$-grading should be regarded as a replacement for an absent linear structure.* The trick is that, differently from the vector bundle case, one should allow to count degrees in different variables with possibly different weights (a situation well known in algebraic topology). We systematically develop this approach in the next sections.

4. Graded manifolds

DEFINITION 4.1. Consider a domain $U \subset \mathbb{R}^{n|m}$ with coordinates x^a (some even, some odd). Assign to them *weights*: $w(x^a) = w_a \in \mathbb{Z}$. Consider the algebra of all smooth functions of x^a that are polynomial in coordinates of nonzero weight. Denote it A. It is a $\mathbb{Z}$-graded algebra (as well as $\mathbb{Z}_2$-graded; these two gradings are independent). We assume that U is cylindrical in the directions of even variables of nonzero weight. There are naturally defined morphisms of pairs (U, A) (homomorphisms of algebras in particular must respect weight). A *graded manifold* is a locally ringed space modeled on such pairs. *Morphisms* of graded manifolds preserve weight of functions.

Simply speaking, a graded manifold M is a supermanifold with a privileged class of atlases in which particular coordinates are assigned numbers called "weight" so that the changes of coordinates respect these weights.

On every graded manifold there is an action of the *scaling transformations* (group $\mathbb{R}^*$ under multiplication). In coordinates, $(x^a) \mapsto (\lambda^{w_a} x^a)$, $\lambda \in \mathbb{R}^*$.

REMARK 4.1. It is possible to consider non-integer weights as well. The difference between coordinates of zero and nonzero weight is similar to the distinction between physical quantities that in some system of units are dimensionless and those that are not (e.g., measured in centimeters). It is possible to consider a function like sine or logarithm of a dimensionless quantity, but not of the one that is measured in centimeters or grams, for which only homogeneous algebraic operations are permitted.

The idea of a "graded manifold" is, of course, very natural. Formal "$\mathbb{Z}$-graded manifolds" were used by Kontsevich in [**8**], however, his $\mathbb{Z}$-grading simply underlies $\mathbb{Z}_2$-grading. Our definition is very close to the notion of "N-manifolds" suggested by Ševera in [**33**]. However, Ševera also couples $\mathbb{Z}_2$-grading and $\mathbb{Z}$-grading. The point in Definition 4.1 is that parity and weight should be considered completely independently.

EXAMPLES. 1. An arbitrary supermanifold: assign zero weight to all coordinates. 2. A vector space: assign the same (arbitrarily chosen) number N as weight to all linear coordinates. 3. A vector bundle: assign N for all linear fiber coordinates (even or odd) and assign zero to all base coordinates. 4. A double vector bundle, like TE or T^*E for a vector bundle E.

For vector spaces and vector bundles $N = 1$ is a standard choice. In general, for a vector bundle E we can write $E(N)$ for the choice N, so $E = E(1)$. Denote $E(N)^* := E^*(-N)$. Then the natural pairing has zero weight.

THEOREM 4.1. *Suppose M is a graded manifold. Then the manifolds TM, T^*M, $TM\Pi$, ΠT^*M are bigraded. One grading is the standard vector bundle grading, the other is the induced weight.*

PROOF. We can lift the scaling transformations to the bundles TM, T^*M, $TM\Pi$, ΠT^*M. If on M we have $w(x^a) = w_a$, then for the natural coordinates in the fibers of TM, T^*M, $TM\Pi$, ΠT^*M we obtain, respectively, $w(v^a) = w_a$, $w(p_a) = -w_a$, $w(dx^a) = w_a$, $w(x^*_a) = -w_a$. It can be checked directly that coordinate changes respect these weights. □

Various total gradings can be constructed as linear combinations of these two.

Notice that on graded manifolds all geometric objects assume weights. In particular, for a vector field $X \in \mathrm{Vect}(M)$, if $X = X^a(x)\partial_a$, then $w(X) = w(X^a) - w_a$. Thus the induced weight on T^*M of the corresponding Hamiltonian $p(X) = X^a p_a$ equals the weight of X on M. The canonical Poisson bracket on T^*M has induced weight zero, because the derivatives w.r.t. x^a and p_a have opposite weights, which mutually cancel. That means that $w(\{f,g\}) = w(f) + w(g)$. It follows that the weight of the Hamiltonian vector field X_f on T^*M is the same as that of a function f. On the other hand, the fiberwise degree of the bracket on T^*M is -1. Similar facts hold for ΠT^*M and the canonical Schouten bracket.

REMARK 4.2. The topological structure of graded manifolds is an interesting question. Precisely: what kind of restrictions for a (super)manifold poses the existence of atlases with coordinate changes preserving weight.

First, it is rather obvious that if all weights are nonnegative and no "external constants" of nonzero weight (see below) are allowed, then under mild assumptions, every graded manifold can be decomposed into a tower of fibrations of the form

$$M = M_N \to M_{N-1} \cdots \to M_2 \to M_1 \to M_0 \tag{4.1}$$

where coordinates on M_0 have zero weight, $M_1 \to M_0$ is a vector bundle, and all other fibrations $M_{k+1} \to M_k$ are "affine bundles", in the sense that changes of variables are linear in fiber coordinates plus additive terms of appropriate weight. (We group all coordinates by decreasing weights $w_N > w_{N-1} > \ldots > w_2 > w_1 > w_0 = 0$; at step N coordinates in the fiber have weight w_N and coordinates on the base M_{N-1} have smaller weights, etc.)

Second, similarly to supermanifolds (for which we need external odd parameters or "odd constants" to be able to consider universal families, etc.), for graded manifolds we might need, in addition, to use external parameters ("constants") with assigned nonzero weights. They can enter formulas that would otherwise be not homogeneous. The role of such constants is similar to "universal constants" in physics or simply to a choice of physical units. An example of situation where this is necessary is constructing metric (of zero weight) on the total space of a vector bundle, from a metric on the base and a fiberwise metric in the bundle (which has weight -2 for the standard choice).

5. Generalized Lie bialgebroids and their doubles

Consider a graded QS-manifold M. Suppose that the homological field $\hat{Q}$ has weight q and the Schouten bracket $\{\ ,\ \}_S$ has weight s. To them correspond odd Hamiltonians (functions on T^*M) $Q = p_{\hat{Q}}$ (fiberwise degree 1) and S (fiberwise degree 2) of weights q and s, respectively. To functions Q and S correspond Hamiltonian vector fields X_Q and X_S on T^*M. It is convenient to arrange the information about gradings in a table:

	$\hat{Q}$	$\{\ ,\ \}_S$	Q	S	$\{\ ,\ \}$	X_Q	X_S
weight on M	q	s	–	–	–	–	–
w = induced weight on T^*M	–	–	q	s	0	q	s
d = fiberwise degree	–	–	1	2	-1	0	1
$w + (q-s)d$	–	–	$2q-s$	$2q-s$	$-q+s$	q	q

The last row follows by a direct calculation.

THEOREM 5.1. *Define the total weight on T^*M as $w+(q-s)d$, where w is the induced weight and d the fiberwise degree. Define the vector field $\hat{Q}_D = X_Q + \lambda X_S \in \mathrm{Vect}(T^*M)$ (where $\lambda \in \mathbb{R}$ is arbitrary, so it is a pencil of fields). Then T^*M with $\hat{Q}_D$ is a graded Q-manifold; the homological vector field $\hat{Q}_D$ is of weight q, the same as the weight of $\hat{Q}$ on M.*

PROOF. Consider the Hamiltonian $Q_D = Q + \lambda S$. (Notice that it is odd, as it should be.) We have: $\{Q,Q\} = p([\hat{Q},\hat{Q}]) = 0$ (as $\hat{Q}^2 = 0$), $\{S,S\} = 0$ (because S defines a Schouten bracket), and $\{Q,S\} = L_{\hat{Q}}S = 0$ (because $\hat{Q}$ is a derivation of the bracket). Thus $\{Q_D,Q_D\} = 0$. Hence, $[\hat{Q}_D,\hat{Q}_D] = 2\hat{Q}_D^2 = 0$. The part concerning weights follows from the table above. □

Suppose $\hat{Q} = Q^a(x)\partial/\partial x^a$ and $S = (1/2)S^{ab}(x)p_bp_a$. Then the explicit formula for $\hat{Q}_D$ is

$$\hat{Q}_D = \left(Q^a + p_bS^{ba}\right)\frac{\partial}{\partial x^a} - (-1)^{\tilde{a}}\left(\frac{\partial Q^b}{\partial x^a}p_b + \frac{1}{2}\frac{\partial S^{bc}}{\partial x^a}p_cp_b\right)\frac{\partial}{\partial p_a} \tag{5.1}$$

(we set for simplicity $\lambda = 1$).

We denote T^*M with the described graded Q-manifold structure by DM and call it the *double* of a graded QS-manifold M. We can consider a graded QS-manifold M as a generalized Lie bialgebroid (more precisely, it is a generalized antibialgebroid, i.e., a generalization of ΠE for a bialgebroid E). The case $q = w(\hat{Q}) = +1$, $s = w(S) = -1$ is the closest to the usual Lie bialgebroids.

EXAMPLE 5.1. Let $M = \Pi E$ for a Lie bialgebroid $E \to X$. Obviously, DM as a Q-manifold coincides with Roytenberg's "Drinfeld's double" of the bialgebroid E. Let us calculate weights according to the recipe of Theorem 5.1. For $M = \Pi E$ we have the standard vector bundle weights: $w(x^a) := 0$, $w(\xi^i) := 1$ (in the notation of Section 3). The homological field has weight 1, the Schouten bracket weight -1 (see (3.5-3.6)). Then on $DM = T^*\Pi E$ we get the induced weights $w(p_a) = 0$, $w(\pi_i) = -1$ and the fiberwise degree $d(p_a) = 1$, $d(\pi_i) = 1$. Hence, our general recipe for the total weight yields $w + (1-(-1))d = w + 2d = \#\xi - \#\pi + 2(\#p + \#\pi) = \#\xi + \#\pi + 2\#p$, which is exactly (3.17).

The construction of the double *mutatis mutandis* carries over to QP-manifolds. Let M be a graded QP-manifold with the homological field $\hat{Q}$ of weight q and the Poisson bracket $\{\ ,\ \}_P$ of weight p. To them correspond even multivector fields $Q' = \theta(\hat{Q}) = -Q^a(x)x_a^*$ (fiberwise degree 1) and $P = (1/2)P^{ab}(x)x_b^*x_a^*$ (fiberwise degree 2) of weights q and p, respectively. Consider the supermanifold ΠT^*M endowed with the canonical Schouten bracket. We can again draw a table:

	$\hat{Q}$	$\{\ ,\ \}_P$	Q'	P	$\{\ ,\ \}$	$X_{Q'}$	X_P
weight on M	q	p	–	–	–	–	–
w = induced weight on ΠT^*M	–	–	q	p	0	q	p
d = fiberwise degree	–	–	1	2	-1	0	1
$w + (q-p)d$	–	–	$2q-p$	$2q-p$	$-q+p$	q	q

Here $X_{Q'} := \{Q',\ \}$ and $X_P := \{P,\ \}$ stand for the odd Hamiltonian vector fields corresponding to the even functions $Q', P \in C^\infty(\Pi T^*M)$.

THEOREM 5.2. *Define the total weight on ΠT^*M as $w + (q-p)d$, where w is the induced weight and d the fiberwise degree. Define the vector field $\hat{Q}_D = X_{Q'} + \lambda X_P \in \mathrm{Vect}(\Pi T^*M)$ (where $\lambda \in \mathbb{R}$ is arbitrary). Then ΠT^*M with $\hat{Q}_D$ is a graded Q-manifold, and the homological vector field $\hat{Q}_D$ is of weight q, which is the same as the weight of $\hat{Q}$ on M.*

PROOF. Similar to the proof of Theorem 5.1. Consider the even function $Q'_D = Q' + \lambda P \in C^\infty(\Pi T^*M)$. We have: $\{Q', Q'\} = \theta([\hat{Q}, \hat{Q}]) = 0$ (as $\hat{Q}^2 = 0$), $\{P, P\} = 0$ (because P defines a Poisson bracket), and $\{Q', P\} = L_{\hat{Q}} P = 0$ (because it is equivalent to $\hat{Q}$ being a derivation of the bracket $\{\ ,\ \}_P$, by the virtue of the non-degeneracy of the canonical Schouten bracket, cf. the proof of Theorems 2.4 and 3.3). Thus $\{Q'_D, Q'_D\} = 0$. Hence, $[\hat{Q}_D, \hat{Q}_D] = 2\hat{Q}_D^2 = 0$. The part concerning weights follows from the table. □

We call the graded Q-manifold ΠT^*M the *double* of the graded QP-manifold M and denote it $DM := \Pi T^*M$.

The double of a graded QS-manifold is automatically a QP-manifold (with respect to the canonical symplectic structure). In the same way, the double of a graded QP-manifold is automatically a QS-manifold, with respect to the canonical odd symplectic structure. This corresponds to the existence of a natural ad-invariant inner product on Drinfeld's double $\mathfrak{d}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$. The following theorem gives partial analogs of Drinfeld's and Manin's theorems.

THEOREM 5.3. *Let M be a graded QS-manifold with the homological field $\hat{Q}$ and the Schouten bracket $\{\ ,\ \}_S$. The homological vector field $\hat{Q}_D \in \mathrm{Vect}(DM)$ on the double $DM = T^*M$ has the following properties:*

(1) *The canonical symplectic structure on T^*M is $\hat{Q}_D$-invariant,*
(2) *$\hat{Q}_D$ is tangent to $M \subset DM$, and $\hat{Q}_{D|M} = \hat{Q}$,*
(3) *For arbitrary functions $f, g \in C^\infty(M)$ the Poisson bracket $\{f, \hat{Q}_D g\}$ also belongs to $C^\infty(M)$, and $\{f, \hat{Q}_D g\} = \{f, g\}_S$,*

and it is uniquely defined by these properties. Conversely, if for a graded manifold M, on the cotangent bundle T^*M there is a homological vector field $\hat{Z}$ such that it preserves the canonical Poisson bracket, is tangent to $M \subset T^*M$, and for arbitrary functions $f, g \in C^\infty(M)$ the bracket $\{f, \hat{Z} g\}$ is also a function on M, then M is a graded QS-manifold w.r.t. the vector field $\hat{Q} := \hat{Z}_{|M}$ and the odd bracket defined as $\{f, g\}_S := \{f, \hat{Z} g\}$.

PROOF. Consider $\hat{Q}_D$ on the double DM. The statement (1) is clear. The statement (2) immediately follows from the coordinate formula (5.1). To obtain (3), notice that $\hat{Q}_D g = \hat{Q} g + \{S, g\}$, hence, because $\hat{Q} g \in C^\infty(M)$, it follows that $\{f, \hat{Q}_D\} = \{f, \{S, g\}\} = \{f, g\}_S$, for arbitrary functions $f, g \in C^\infty(M)$). Now suppose that for an arbitrary homological field $\hat{Z} \in \mathrm{Vect}(DM)$ the properties (1),(2),(3) hold. Then, by (1), we conclude that $\hat{Z}$ is Hamiltonian, with some odd Hamiltonian $Z \in C^\infty(DM)$, where

$$Z = Z_0(x) + Z^a(x) p_a + \frac{1}{2} Z^{ab}(x)\, p_b p_a + \frac{1}{3!} Z^{abc}(x)\, p_c p_b p_a \ldots, \tag{5.2}$$

hence

$$\hat{Z} = \left(Z^a(x) + p_b Z^{ba}(x) + \frac{1}{2} p_b p_c Z^{cba}(x) + \dots \right) \frac{\partial}{\partial x^a} - (-1)^{\tilde{a}} \left(\frac{\partial Z_0}{\partial x^a} + \frac{\partial Z^b}{\partial x^a} p_b + \frac{1}{2} \frac{\partial Z^{bc}}{\partial x^a} p_c p_b + \dots \right) \frac{\partial}{\partial p_a}. \tag{5.3}$$

From the condition (2) we conclude that $\partial Z_0/\partial x^a = 0$, hence w.l.o.g. Z_0 can be set to zero, and that $Z^a = Q^a$. Taking $\{f, \hat{Z}g\} = \{f, \{Z, g\}\}$ for arbitrary functions on M and applying the condition (3), we deduce that no terms of order > 2 can be present in (5.2) and that $Z^{ba} = S^{ba}$. This proves the uniqueness of $\hat{Q}_D$. Similar argument is applicable to an arbitrary homological field $\hat{Z} \in C^\infty(T^*M)$ with the stated properties (without assuming a given QS-structure on M). We get a field $\hat{Q} := \hat{Z}_{|M} \in \mathrm{Vect}(M)$ and a bracket $\{f, g\}_S := \{f, \hat{Z}g\}$. That they make a QS-structure follows directly. □

Similar theorem holds for QP-manifolds.

Now, following the example of Lie bialgebras, we want to provide the doubles of QS- and QP-manifolds with brackets of the *same* parity as the original one. E.g., for a QS-manifold M, we want a $\hat{Q}_D$-invariant Schouten bracket on $DM = T^*M$. To this end, recall the construction of the Schouten bracket on $T^*\Pi\mathfrak{g}$ (see Theorem 2.6). We shall try to mimick it.

Let x^a be local coordinates on M. Let us denote the corresponding fiber coordinates in T^*M by y_a now (the letter p is reserved for a different use). Consider the second cotangent bundle T^*T^*M. Let p_a and q^a stand for fiber coordinates in $T^*(T^*M)$ (the conjugate momenta for x^a and y_a, respectively). The changes of coordinates have the following form (cf. equation (7.5) in Appendix):

$$\begin{cases} x^a = x^a(x') \\ y_a = J_a{}^{a'}(x')\, y_{a'} \\ p_a = \dfrac{\partial x^{a'}}{\partial x^a} p_{a'} - (-1)^{\tilde{a}\tilde{b}+\tilde{c}'} q^{c'} J_{c'}{}^b \dfrac{\partial J_b{}^{a'}}{\partial x^a} y_{a'} \\ (-1)^{\tilde{a}} q^a = (-1)^{\tilde{a}'} q^{a'} J_{a'}{}^a \end{cases} \tag{5.4}$$

Here $J_a{}^{a'}$ stands for the Jacobi matrix. Thus, $p_a q^a = (-1)^{\tilde{a}} q^a p_a$ is not invariant. However, if we are given a linear connection on M, so that for a local section of T^*M the covariant derivative is

$$\nabla_a \alpha_b = \partial_a \alpha_b - \Gamma_{ab}{}^c \alpha_c, \tag{5.5}$$

then the following "long" momentum

$$P_a := p_a + (-1)^{\tilde{b}(\tilde{a}+1)} q^b \Gamma_{ab}{}^c y_c = p_a + \Gamma_{ab}{}^c y_c q^b \tag{5.6}$$

transforms as

$$P_a = \frac{\partial x^{a'}}{\partial x^a} P_{a'}, \tag{5.7}$$

by a direct computation. Hence the expression $P_a q^a \in C^\infty(T^*T^*M)$ is invariant. Denote it by r:

$$r = P_a q^a = p_a q^a + \Gamma_{ab}{}^c y_c q^b q^a. \tag{5.8}$$

Following the example of Drinfeld's double for Lie bialgebras, we want to define the Schouten tensor for DM as $S_D := (1/2)\{Q_D, r\}$.

In the following *we assume that there is a connection of zero weight* at our disposal. Then $w(\Gamma^c_{ab}) = w_c - w_a - w_b$. Hence, $w(\Gamma_{ab}{}^c y_c q^b) = w_c - w_a - w_b - w_c + w_b = -w_a = w(p_a)$ (w everywhere stands for weight on M and the induced weight on natural bundles), and the long momentum P_a is homogeneous of weight $-w_a$. We shall also use the total weight $W := w + (q-s)d$ on DM and the corresponding induced weight on T^*DM. Notice that $W(x^a) = w_a$, $W(y_a) = -w_a + q - s$, $W(p_a) = -w_a$, $W(q^a) = w_a - q + s$.

PROPOSITION 5.1. *The function $S_D = (1/2)\{Q_D, r\} \in C^\infty(T^*DM)$ is quadratic in momenta p_a, q^a and has weight $W(S_D) = s$.*

PROOF. Because r is quadratic, Q_D is linear and the canonical bracket has degree -1, the function S_D is quadratic. Now, let us calculate weight. The weight of Q_D is the same as that of $\hat{Q}_D$, $W(Q_D) = q$, and the weight of r is $W(r) = -w_a + w_a - q + s = -q + s$. Hence, because the canonical bracket has zero weight, $W(S_D) = W(Q_D) + W(r) = q - q + s = s$. □

Thus, the odd Hamiltonian S_D defines on the double DM an "almost" Schouten bracket of weight s automatically compatible with $\hat{Q}_D$. We shall give an explicit formula for it. To this end, we may consider x^a, y_a, P_a, q^a as convenient coordinates on T^*DM (though they are not Darboux). For a function $f \in C^\infty(DM)$ introduce a "covariant partial derivative"

$$\nabla_a f := \frac{\partial f}{\partial x^a} + \Gamma^c_{ab} y_c \frac{\partial f}{\partial y_b}. \tag{5.9}$$

It transforms as a component of a covector on M. Notice that the partial derivative $\partial f/\partial y_a$ transforms as a component of a vector. We also adopt convention (usual in tensor calculus) that that any operator nabla applied to a quantity with tensor indices automatically incorporates extra Christoffel symbols.

LEMMA 5.1. *For an arbitrary function $f \in C^\infty(DM)$, the Poisson bracket of $p(X_f)$ with $r = P_a q^a$ is given by the formula*

$$\{p(X_f), r\} = (-1)^{\tilde{f}(\tilde{a}+\tilde{b})} \left(\nabla_a \nabla_b f - (-1)^{\tilde{b}\tilde{c}} R^k_{acb} y_k \frac{\partial f}{\partial y_c} \right) q^b q^a - (-1)^{(\tilde{f}+1)(\tilde{a}+\tilde{b})} \frac{\partial^2 f}{\partial y_a \partial y_b} P_b P_a. \tag{5.10}$$

Here R^k_{acb} is the Riemann tensor; the outer ∇_a contains an extra Christoffel symbol for the tensor index of the inner ∇_b.

Applying this lemma to the functions $Q = Q^a(x) y_a$ and $S = (1/2) S^{ab}(x) y_b y_a$, we get

THEOREM 5.4. *A $\hat{Q}_D$-invariant almost Schouten bracket on the double DM of a graded QS-manifold M is given by the Hamiltonian $S_D = (1/2)\{Q_D, r\}$ which*

has the following explicit expression

$$S_D = \frac{1}{2}(-1)^{\tilde{a}+\tilde{b}}\Bigg(\Big(\nabla_a\nabla_b Q^k - (-1)^{\tilde{b}\tilde{c}+\tilde{k}(\tilde{c}+1)}R^k_{acb}Q^c\Big)y_k + \Big(\frac{1}{2}\nabla_a\nabla_b S^{kl} - (-1)^{\tilde{b}\tilde{c}+(\tilde{k}+\tilde{l})(\tilde{k}+\tilde{c})}R^l_{acb}S^{kc}\Big)y_l y_k\Bigg)q^b q^a - \frac{1}{2}S^{ab}(x)P_bP_a. \tag{5.11}$$

Here R^k_{acb} is the Riemann tensor and nabla denotes the covariant derivative of tensor fields on the graded manifold M, w.r.t. the chosen linear connection.

I skip the proofs of Lemma 5.1 and Theorem 5.4. Notice that these statements are nonlinear analogs of Lemma 2.1 and formula (2.20) for Lie bialgebras. The difference with the Lie bialgebra case is due to two independent reasons. One is a possible non-flatness of M, hence the necessity to use a connection and the occurrence of curvature as a result. The other difference is more substantial. In a flat situation one can replace covariant derivatives by ordinary partial derivatives. In Drinfeld's case, when Q^k is quadratic in coordinates, the first term in (5.11) would simply recover Q^k. For a general nonlinear Q^k, it would not be so. This is a hint that we should not expect, generally, for the almost Schouten bracket given by S_D to obey Jacobi.

EXAMPLE 5.2. Consider $M = \mathbb{R}^{1|3}$ as a graded manifold, with an even coordinate x and odd coordinates ξ^i where we assign weights as $w(x) = 2$ and $w(\xi^i) = 1$. Consider

$$\hat{Q} = \big(x\xi^1 + \xi^1\xi^2\xi^3\big)\frac{\partial}{\partial x} + \xi^1\xi^3\frac{\partial}{\partial\xi^1} + \big(x + \xi^1\xi^2\big)\frac{\partial}{\partial\xi^2}. \tag{5.12}$$

One can directly check that $\hat{Q}^2 = 0$. The homological field (5.12) specifies on $\mathbb{R}^{3|1}$ a structure of an L_∞-algebra, which is in fact homotopy equivalent to a differential Lie superalgebra. We want to consider M as a nonlinear analog of a Lie algebroid. Notice that $w(\hat{Q}) = +1$. Take the identically zero Schouten bracket on M. Hence we have a graded QS-manifold which is an analog of a Lie algebra considered as a bialgebra (with the zero cobracket). The double of M is $DM = T^*M$ with the Hamiltonian homological vector field of weight $+1$

$$\hat{Q}_D = \big(x\xi^1 + \xi^1\xi^2\xi^3\big)\frac{\partial}{\partial x} + \xi^1\xi^3\frac{\partial}{\partial\xi^1} + \big(x + \xi^1\xi^2\big)\frac{\partial}{\partial\xi^2} - \big(\xi^1 y + \eta_2\big)\frac{\partial}{\partial y} + \Big(\big(x + \xi^2\xi^3\big)y + \xi^3\eta_1 + \xi^2\eta_2\Big)\frac{\partial}{\partial\eta_1} - \big(\xi^1\xi^3 y + \xi^1\eta_2\big)\frac{\partial}{\partial\eta_2} + \big(\xi^1\xi^2 y - \xi^1\eta_1\big)\frac{\partial}{\partial\eta_3},$$

where y and η_i are the conjugate momenta for x and ξ^i. The "total weights" are: $W(x) = 2$, $W(\xi^i) = 1$, $W(y) = 0$, $W(\eta_i) = 1$ (we assumed that $w(S) = -1$). $\hat{Q}_D$ is just the Lie derivative along $\hat{Q}$. (In the linear case, this corresponds to a semidirect product Lie algebra structure in $\mathfrak{g} \oplus \mathfrak{g}^*$ given by the coadjoint representation.) We can construct a $\hat{Q}_D$-invariant almost Schouten bracket on DM as outlined above. Let the momenta conjugate to x, ξ^i, y, η_i on $T^*DM = T^*T^*M$ be denoted as p, π_i,

q, $\varkappa^i$, respectively. Take a flat connection on M so that the covariant derivatives w.r.t. x, ξ^i are simply the partial derivatives. Then $r = pq + \pi_i \varkappa^i$ and either directly calculating or using Theorem 5.4 we obtain $S_D = (1/2)\{Q_D, r\}$ as

$$S_D = -y\varkappa^1 q - \left(\xi^3 y + \eta_2\right)\varkappa^2\varkappa^1 + \left(\xi^2 y - \eta_1\right)\varkappa^3\varkappa^1 - \xi^1 y\varkappa^3\varkappa^2,$$

which gives the following invariant almost Schouten brackets:

$$\begin{aligned}
\{y, \eta_1\}_{S_D} &= y \\
\{\eta_1, \eta_2\}_{S_D} &= -\xi^3 y - \eta_2 \\
\{\eta_1, \eta_3\}_{S_D} &= -\xi^3 y - \eta_1 \\
\{\eta_2, \eta_3\}_{S_D} &= -\xi^1 y.
\end{aligned}$$

Notice that they are homogeneous of weight -1. As expected, the only non-zero brackets are between the "dual" coordinates y, η_i (in the Lie algebra case we would recover the Lie–Poisson bracket on $\mathfrak{g}^*$). However, they depend on variables ξ^i as parameters, due to the nonlinearity of the original field (5.12). The Jacobi identity fails, as $\{S_D, S_D\}_{T^*T^*M} = 2\left((y\xi^3 + \eta_2)\varkappa^1\varkappa^2\varkappa^3 - yq\varkappa^1\varkappa^3\right) \neq 0$, by a direct calculation. This can be attributed either to a bad choice of a connection or to the fact that another manifestation of our graded Q-manifold M is a strong homotopy Lie algebra — hence it would be plausible to look for "homotopy" Schouten brackets on DM. In fact, both considerations can be related. As mentioned above, the L_∞-algebra defined by $\hat{Q}$ is homotopy equivalent to an ordinary Lie superalgebra; i.e., the cubic term in $\hat{Q}$ can be killed by a nonlinear transformation. There is no problem to construct a genuine bracket on the dual of a Lie superalgebra; it should give a "homotopy" bracket on a homotopy equivalent object. Or, this amounts to a different choice of a flat connection, in "linear" coordinates on M corresponding to the Lie algebra structure rather than in x, ξ^i.

The toy example above illustrates the general situation (except that in general we do not have a "linear" reference object). It points to a link between the nonlinearity and homotopy . It is probable that the double for a nonlinear QS- or QP-manifold should be equipped with something like a "homotopy Schouten bracket" rather than an ordinary one. This should be explored further.

6. Odd Lie bialgebras and the odd double

In this section we return to the linear situation and specialize our approach for odd Lie bialgebras, which were defined in Section 2. Notice that an odd bialgebra structure can be nontrivial (with a nontrivial odd cobracket) only in the super case, for $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_1 \neq 0$.

Let $\mathfrak{g}$ be an odd Lie bialgebra. We describe its structure using the supermanifold $\Pi\mathfrak{g}$. Let the homological vector field $\hat{Q} = (1/2)\xi^j\xi^i Q_{ij}^k \partial/\partial\xi^k$ correspond to the Lie superalgebra structure in $\mathfrak{g}$ and let the Poisson tensor $P = (1/2)\xi^k P_k^{ij} \xi_j^* \xi_i^*$ correspond to the Lie superalgebra structure in $\Pi\mathfrak{g}^*$. Then the Lie brackets in $\mathfrak{g}$ are given by the formula

$$[e_i, e_j] = (-1)^{\tilde{\jmath}} Q_{ij}^k e_k, \tag{6.1}$$

and the brackets in $\Pi\mathfrak{g}^*$ are given by

$$[\varepsilon^i, \varepsilon^j] = (-1)^{\tilde{\imath}+1} \varepsilon^k P_k^{ij}. \tag{6.2}$$

Here $\varepsilon^i = \Pi e^i$ where the basis e^i is right-dual to e_j. The last formula follows from the definition of the Lie–Poisson bracket (the Poisson bracket of the elements of $\Pi\mathfrak{g}^*$ considered as linear functions on $\Pi\mathfrak{g}$ equals their Lie bracket) and the formulas from Section 1. It also coincides with the one given by the duality argument, see Example 7.1 in the Appendix. (Notice that $\xi^i = \varepsilon^i$, as well as $x_i = e_i$.) It is not difficult to see that the homological vector field on $\mathfrak{g}^* = \Pi\Pi\mathfrak{g}^*$ defining the bracket (6.2) is $\hat{Q}^* = (1/2)(-1)^{\tilde{k}+1} P_k^{ij} x_j x_i \partial/\partial x_k$.

The odd bialgebra conditions for $\mathfrak{g}$ are

$$\{Q, Q\} = 0, \quad \{P, P\} = 0, \quad \{Q, P\} = 0, \tag{6.3}$$

with the canonical Schouten brackets in $\Pi T^*\Pi\mathfrak{g}$, where P is the Poisson tensor and $Q = \theta(\hat{Q}) = -(1/2)\xi^j \xi^i Q_{ij}^k \xi_k^* \in C^\infty(\Pi T^*\Pi\mathfrak{g})$ is the even antivector field corresponding to the homological field $\hat{Q}$. (There should be no confusion with Hamiltonians used in the previous sections.)

LEMMA 6.1. *For every Lie superalgebra $\mathfrak{g}$, on its antidual space $\Pi\mathfrak{g}^*$ there is a naturally defined odd linear map ("odd cobracket")*

$$\delta\colon \Pi\mathfrak{g}^* \to S^2(\Pi\mathfrak{g}^*) \subset \Pi\mathfrak{g}^* \otimes \Pi\mathfrak{g}^*. \tag{6.4}$$

If the Lie bracket is given by (6.1), *then the odd cobracket for it is given by the formula*

$$\delta(\varepsilon^k) = -\varepsilon^j \otimes \varepsilon^i Q_{ij}^k = -\varepsilon^j \varepsilon^i Q_{ij}^k. \tag{6.5}$$

Here $\varepsilon_j \varepsilon_i = (1/2)(\varepsilon_j \otimes \varepsilon_i + (-1)^{(\tilde{\imath}+1)(\tilde{\jmath}+1)} \varepsilon_i \otimes \varepsilon_j)$ is the symmetric product.

PROOF. The map $\delta\colon \Pi\mathfrak{g}^* \to \Pi\mathfrak{g}^* \otimes \Pi\mathfrak{g}^*$ for an arbitrary Lie superalgebra $\mathfrak{g}$ is defined as follows. If we consider the Lie bracket as the linear map $\delta^*\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, then the usual adjoint gives the map $\delta'\colon \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^*$, and δ is the composition of δ' with the natural isomorphism $\mathfrak{g}^* \otimes \mathfrak{g}^* \to \Pi\mathfrak{g}^* \otimes \Pi\mathfrak{g}^*$ and the natural (odd) isomorphism $\Pi\mathfrak{g}^* \to \mathfrak{g}^*$ (left multiplication by Π). In terms of bases we have:

$$\langle e_i \otimes e_j, \delta'(e^k)\rangle = \langle [e_i, e_j], e^k\rangle = \langle (-1)^{\tilde{\jmath}} Q_{ij}^l e_l, \varepsilon^k\rangle = (-1)^{\tilde{\jmath}} Q_{ij}^k,$$

hence $\delta'(e^k) = (-1)^{\tilde{\jmath}+\tilde{\imath}\tilde{\jmath}} e^i \otimes e^j Q_{ij}^k = (-1)^{\tilde{\imath}+1} e^i \otimes e^j Q_{ji}^k$. (Notice that for the basis e^i right-dual to e_i, $\langle e_i \otimes e_j, e^k \otimes e^l\rangle = (-1)^{\tilde{\imath}\tilde{\jmath}} \delta_i{}^k \delta_j{}^l$.) Now, we have $\varepsilon^k \mapsto \Pi\varepsilon^k = \Pi\Pi e^k = e^k$, and we use the canonical identification $\varepsilon^i \otimes \varepsilon^j = \Pi e^i \otimes \Pi e^j$ with $(-1)^{\tilde{\imath}} e^i \otimes e^j$. Thus, for the composition, we get $\delta\colon \varepsilon^k \mapsto e^k \mapsto (-1)^{\tilde{\imath}+1} e^i \otimes e^j Q_{ji}^k \mapsto -\varepsilon^i \otimes \varepsilon^j Q_{ji}^k$. Or:

$$\delta(\varepsilon^k) = -\varepsilon^j \otimes \varepsilon^i Q_{ij}^k = -\frac{1}{2}\left(\varepsilon^j \otimes \varepsilon^i Q_{ij}^k + \varepsilon^i \otimes \varepsilon^j Q_{ji}^k\right) =$$
$$-\frac{1}{2}\left(\varepsilon^j \otimes \varepsilon^i + \varepsilon^i \otimes \varepsilon^j (-1)^{(\tilde{\imath}+1)(\tilde{\jmath}+1)}\right) Q_{ij}^k = -\varepsilon^j \varepsilon^i Q_{ij}^k.$$

In other words, $\delta = -2\hat{Q}$ restricted to linear functions (with $\varepsilon^k = \xi^k$). □

PROPOSITION 6.1. *If $\mathfrak{g}$ is an odd bialgebra and the Lie bracket in $\Pi\mathfrak{g}^*$ is given by formula* (6.2), *then the odd cobracket on $\mathfrak{g}$ is an odd linear map*

$$\delta\colon \mathfrak{g} \to S^2(\mathfrak{g}) \subset \mathfrak{g} \otimes \mathfrak{g} \tag{6.6}$$

defined by the formula

$$\delta(e_k) = (-1)^{\tilde{k}} P_k^{ij} e_j \otimes e_i = (-1)^{\tilde{k}} P_k^{ij} e_j e_i. \tag{6.7}$$

PROOF. Immediately follows from Lemma 6.1. For an odd bialgebra, to obtain formula (6.7) for the odd cobracket in $\mathfrak{g}$ we simply use the remark at the end of proof for Lemma 6.1 together with the above formula for the vector field $\hat{Q}^*$. (Notice that $e_i = x_i$.) □

EXAMPLE 6.1. Consider a 1|1-dimensional Lie superalgebra $\mathfrak{g}$ with a basis $e \in \mathfrak{g}_0$, $\varepsilon \in \mathfrak{g}_1$ such that $[\varepsilon, \varepsilon] = 2e$ (the "supersymmetry algebra"). There is an odd inner product in $\mathfrak{g}$ defined by the condition $(e, \varepsilon) = -(\varepsilon, e) = 1$. One can check that this product is $\operatorname{ad}\mathfrak{g}$-invariant. Moreover, we can use this product to obtain a cobracket on $\mathfrak{g}$ (as odd-dual to the given Lie bracket). We have

$$\delta(e) = 0, \tag{6.8}$$

$$\delta(\varepsilon) = e \otimes e. \tag{6.9}$$

The bracket structure in $\mathfrak{g}$ corresponds to the homological field $\hat{Q} = -x^2\,\partial/\partial\xi \in \operatorname{Vect}(\mathfrak{g})$, and the odd cobracket corresponds to the Poisson tensor $P = -x(\xi^*)^2$. Clearly, the Schouten bracket $\{Q, P\}$ is zero. Hence, the cobracket (6.8),(6.9) makes $\mathfrak{g}$ into an odd bialgebra.

Now let us elaborate the construction of the odd double briefly introduced in Section 2. In the sequel we use $\mathfrak{d}(\mathfrak{g}) := \mathfrak{g} \oplus \Pi\mathfrak{g}^*$ as the notation for the odd double. The odd double has a structure of an odd bialgebra defined as follows.

To get the Lie bracket in $\mathfrak{d}(\mathfrak{g})$, consider the sum $Q + P \in C^\infty(\Pi T^*\Pi\mathfrak{g})$ of multivector fields Q and P. By the equations (6.3), it satisfies $\{Q + P, Q + P\} = 0$. Hence the corresponding odd Hamiltonian vector field X_{Q+P} is homological. Calculating by the general formulas from Section 1, we get

$$X_Q = -\frac{1}{2}\,\xi^j\xi^i Q^k_{ji}\frac{\partial}{\partial\xi^k} - (-1)^{\tilde{k}(\tilde{\jmath}+1)}\xi^*_j\xi^i Q^j_{ik}\frac{\partial}{\partial\xi^*_k}, \tag{6.10}$$

$$X_P = (-1)^{\tilde{\jmath}}\xi^*_j\xi^i P^{jk}_i\frac{\partial}{\partial\xi^k} + (-1)^{\tilde{k}(\tilde{\imath}+\tilde{\jmath}+1)}\frac{1}{2}\,\xi^*_j\xi^*_i P^{ij}_k\frac{\partial}{\partial\xi^*_k}. \tag{6.11}$$

Define the vector field $\hat{Q}_D$ on $\Pi T^*\Pi\mathfrak{g}$ as $\hat{Q}_D := -X_{Q+P}$ (notice the minus sign). It is homological; explicitly:

$$\hat{Q}_D = \frac{1}{2}\,\xi^j\xi^i Q^k_{ij}\frac{\partial}{\partial\xi^k} + \xi^*_j\xi^i\left(-(-1)^{\tilde{\jmath}}P^{jk}_i\frac{\partial}{\partial\xi^k} + (-1)^{\tilde{k}(\tilde{\jmath}+1)}Q^j_{ik}\frac{\partial}{\partial\xi^*_k}\right) - \\ - (-1)^{\tilde{k}(\tilde{\imath}+\tilde{\jmath}+1)}\frac{1}{2}\,\xi^*_j\xi^*_i P^{ij}_k\frac{\partial}{\partial\xi^*_k}. \tag{6.12}$$

Conversely, the condition that $\hat{Q}_D^2 = 0$ is equivalent to the condition $\{Q + P, Q + P\} = 0$, i.e., to the odd Lie bialgebra conditions (6.3). The field $\hat{Q}_D$ corresponds to the Lie superalgebra structure in $\mathfrak{d}(\mathfrak{g})$. Notice that $\Pi T^*\Pi\mathfrak{g} \cong \Pi\mathfrak{g} \oplus \mathfrak{g}^* = \Pi(\mathfrak{g} \oplus \Pi\mathfrak{g}^*)$. Here we can identify $x_i = \xi^*_i$. If we translate the vector field (6.12) into the language of Lie brackets, we get the following formulas

$$[e_i, e_j] = (-1)^{\tilde{\jmath}}Q^k_{ij}e_k \tag{6.13}$$

$$[e_i, \varepsilon^j] = P^{jk}_i e_k + \varepsilon^k Q^j_{ki} \tag{6.14}$$

$$[\varepsilon^i, \varepsilon^j] = (-1)^{\tilde{\imath}+1}\varepsilon^k P^{ij}_k \tag{6.15}$$

in the odd double, by a straightforward calculation using formula (2.9)). (Hence $\mathfrak{g} \subset \mathfrak{d}(\mathfrak{g})$ and $\Pi\mathfrak{g}^* \subset \mathfrak{d}(\mathfrak{g})$ are subalgebras.)

To get the odd cobracket in $\mathfrak{d}(\mathfrak{g})$, we need a Poisson bracket on the supermanifold $\Pi\mathfrak{d}(\mathfrak{g}) = \Pi T^*\Pi\mathfrak{g}$. To this end, consider the anticotangent bundle $\Pi T^*\Pi T^*\Pi\mathfrak{g}$. To avoid confusion, on $\Pi T^*\Pi\mathfrak{g}$ rename $\xi_i^* =: x_i$, so the coordinates on $\Pi T^*\Pi\mathfrak{g}$ will be ξ^i, x_i. Denote now by ξ_i^*, x^{*i} the odd conjugate momenta for the coordinates ξ^i, x_i respectively. Clearly, x^{*i} transforms as ξ^i, hence the odd function $\rho = x^{*i}\xi_i^* = \xi_i^* x^{*i}$ on $\Pi T^*\Pi T^*\Pi\mathfrak{g}$ is well-defined. Consider the function P_D defined as $P_D = (1/2)L_{\hat{Q}_D}\rho = (1/2)\{-\theta(\hat{Q}_D), \rho\}$. It follows that P_D is a $\hat{Q}_D$-invariant even bivector field on $\Pi T^*\Pi\mathfrak{g}$. To find it explicitly we can either prove and apply an analog of Lemma 2.1 or just calculate straightforwardly.

PROPOSITION 6.2.

$$P_D = \frac{1}{2}\, x^{*j} x^{*i} Q_{ij}^k x_k + \frac{1}{2}\, \xi^k P_k^{ij} \xi_j^* \xi_i^* \tag{6.16}$$

PROOF. Consider $\rho = \xi_j^* x^{*j}$ and

$$-\theta(\hat{Q}_D) = \frac{1}{2}\, \xi^j \xi^i Q_{ij}^k \xi_k^* + \xi_j^* \xi^i \left(-(-1)^{\tilde{\jmath}} P_i^{jk} \xi_k^* + (-1)^{\tilde{k}(\tilde{\jmath}+1)} Q_{ik}^j x^{*k}\right) - \\ - (-1)^{\tilde{k}(\tilde{\imath}+\tilde{\jmath}+1)} \frac{1}{2}\, \xi_j^* \xi_i^* P_k^{ij} x^{*k}. \tag{6.17}$$

Their Schouten bracket equals

$$\{-\theta(\hat{Q}_D), \rho\} = (-1)^{\tilde{\jmath}} \frac{\partial(-\theta(\hat{Q}_D))}{\partial \xi^j} \frac{\partial \rho}{\partial \xi_j^*} - (-1)^{\tilde{\jmath}} \frac{\partial(-\theta(\hat{Q}_D))}{\partial x_j} \frac{\partial \rho}{\partial x^{*j}} =$$

$$(-1)^{\tilde{\jmath}} \left(\xi^i Q_{ij}^k \xi_k^* + (-1)^{\tilde{\imath}(\tilde{\jmath}+1)} x_i \left(-(-1)^{\tilde{\imath}} P_j^{ik} \xi_k^* + (-1)^{\tilde{k}(\tilde{\imath}+1)} Q_{jk}^i x^{*k}\right)\right) x^{*j} -$$

$$(-1)^{\tilde{\jmath}} \left(\xi^i \left(-(-1)^{\tilde{\jmath}} P_i^{jk} \xi_k^* + (-1)^{\tilde{k}(\tilde{\jmath}+1)} Q_{ik}^j x^{*k}\right) - (-1)^{\tilde{k}(\tilde{\imath}+\tilde{\jmath}+1)} x_i P_k^{ij} x^{*k}\right) \xi_j^* =$$

$$(-1)^{\tilde{\jmath}} \xi^i Q_{ij}^k \xi_k^* x^{*j} - (-1)^{\tilde{\jmath}+\tilde{k}(\tilde{\jmath}+1)} Q_{ik}^j x^{*k} \xi_j^*$$

$$- (-1)^{\tilde{\jmath}+\tilde{\imath}(\tilde{\jmath}+1)+\tilde{\imath}} x_i P_j^{ik} \xi_k^* x^{*j} + (-1)^{\tilde{\jmath}+\tilde{k}(\tilde{\imath}+\tilde{\jmath}+1)} x_i P_k^{ij} x^{*k} \xi_j^* +$$

$$(-1)^{\tilde{\jmath}+\tilde{k}(\tilde{\imath}+1)+\tilde{\imath}(\tilde{\jmath}+1)} x_i Q_{jk}^i x^{*k} x^{*j} + \xi^i P_i^{jk} \xi_k^* \xi_j^*.$$

After exchanging the indices j and k in the second and the fourth terms of the last expression, changing order of factors and calculating signs, it turns out that these terms cancel with the first and the second terms respectively. In the same way, the factors in the fifth term can be rearranged so that it completely cancels signs. Finally:

$$\{-\theta(\hat{Q}_D), \rho\} = x^{*k} x^{*j} Q_{jk}^i x_i + \xi^i P_i^{jk} \xi_k^* \xi_j^*, \tag{6.18}$$

which differs from the required formula only by the notation for the indices. □

COROLLARY 6.1. *P_D is a Poisson tensor on $\Pi\mathfrak{d}(\mathfrak{g})$.*

PROOF. The two terms in (6.16) have the zero canonical Schouten bracket because they do not contain conjugate variables. Each term is a Poisson tensor by itself. Indeed, the second term is the Poisson tensor on $\Pi\mathfrak{g}$. The first term equals $P^* = (1/2)(-1)^{\tilde{\imath}+\tilde{\jmath}} Q_{ij}^k x_k x^{*j} x^{*i}$, the Poisson tensor that gives the Lie–Poisson bracket $\{x_i, x_j\} = (-1)^{\tilde{\jmath}} Q_{ij}^k x_k$ on $\mathfrak{g}^*$ generated by the Lie bracket in $\mathfrak{g}$. □

COROLLARY 6.2. *The nonvanishing Poisson brackets of coordinates on* $\Pi\mathfrak{d}(\mathfrak{g})$ *are:*

$$\{\xi^i, \xi^j\}_{P_D} = (-1)^{\tilde{\imath}+1}\xi^k P_k^{ij}, \quad \{x_i, x_j\}_{P_D} = (-1)^{\tilde{\jmath}} Q_{ij}^k x_k. \tag{6.19}$$

COROLLARY 6.3. *The odd cobracket* $\delta\colon \mathfrak{d}(\mathfrak{g}) \to S^2\mathfrak{d}(\mathfrak{g}) \subset \mathfrak{d}(\mathfrak{g}) \otimes \mathfrak{d}(\mathfrak{g})$ *is given by the formulas:*

$$\delta(e_k) = (-1)^{\tilde{k}} P_k^{ij} e_j e_i, \tag{6.20}$$

$$\delta(\varepsilon^k) = -\varepsilon^j \varepsilon^i Q_{ij}^k. \tag{6.21}$$

Hence, both $\mathfrak{g} \subset \mathfrak{d}(\mathfrak{g})$ *and* $\Pi\mathfrak{g}^* \subset \mathfrak{d}(\mathfrak{g})$ *are sub-bialgebras.*

It is possible to formulate everything entirely in terms of linear spaces $\mathfrak{g}$ and $\Pi\mathfrak{g}^*$. In the corresponding analogs of Theorem 2.1 and Theorem 2.2, there appear odd inner products.

THEOREM 6.1. *Let* $\mathfrak{g}$ *be an odd Lie bialgebra. Then in the vector space* $\mathfrak{d}(\mathfrak{g}) := \mathfrak{g} \oplus \Pi\mathfrak{g}^*$ *there is a natural structure of an odd Lie bialgebra uniquely characterized by the following properties:*

1. $\mathfrak{g}$ *and* $\Pi\mathfrak{g}^*$ *are Lie subalgebras in* $\mathfrak{d}(\mathfrak{g})$,
2. *The natural odd inner product in* $\mathfrak{d}(\mathfrak{g}) = \mathfrak{g} \oplus \Pi\mathfrak{g}^*$ *is* ad*-invariant,*
3. *The odd cobracket* $\delta\colon \mathfrak{d}(\mathfrak{g}) \to S^2\mathfrak{d}(\mathfrak{g}) \subset \mathfrak{d}(\mathfrak{g}) \otimes \mathfrak{d}(\mathfrak{g})$ *is given by the coboundary of the (odd) invariant element* $\rho = e_i\varepsilon^i \in S^2(\mathfrak{g} \oplus \Pi\mathfrak{g}^*)$.

PROOF. The explicit formulas defining the odd bialgebra structure in $\mathfrak{d}(\mathfrak{g})$ are given above: (6.13-6.15) and (6.20-6.21). We shall check uniqueness and the statement concerning the cobracket. Recall that an odd symmetric inner product has the property

$$(u, v) = (-1)^{\tilde{u}+\tilde{v}+\tilde{u}\tilde{v}}(v, u). \tag{6.22}$$

In $\mathfrak{g} \oplus \Pi\mathfrak{g}^*$ the nonvanishing inner products of the basis vectors are $(e_i, \varepsilon^j) = \delta_i{}^j$, $(\varepsilon^j, e_i) = -\delta_i{}^j$. The ad-invariance condition for an odd inner product means that

$$([u, v], w) + (-1)^{\tilde{u}(\tilde{v}+1)}(v, [u, w]) = 0 \tag{6.23}$$

for all u, v, w, or

$$([v, u], w) = (-1)^{\tilde{u}}(v, [u, w]). \tag{6.24}$$

Suppose $[e_i, \varepsilon^j] = f_i^{jk} e_k + \varepsilon^k g_{ki}^j$ with indeterminate coefficients f_i^{jk}, g_{ki}^j. Then $([e_i, \varepsilon^j], \varepsilon^k) = f_i^{jk}$, $(e_k, [e_i, \varepsilon^j]) = g_{ki}^j$. From the ad-invariance we get: $([e_i, \varepsilon^j], \varepsilon^k) = (-1)^{\tilde{\jmath}+1}(e_i, [\varepsilon^j, \varepsilon^k]) = (-1)^{\tilde{\jmath}+1}(e_i, (-1)^{\tilde{\jmath}+1}\varepsilon^l P_l^{jk}) = P_i^{jk}$, and also $(e_k, [e_i, \varepsilon^j]) = (-1)^{\tilde{\imath}}([e_k, e_i], \varepsilon^j) = (-1)^{\tilde{\imath}}((-1)^{\tilde{\imath}} Q_{ki}^l, \varepsilon^j) = Q_{ki}^j$, where we used formulas (6.13) and (6.15) that follow from the condition 1. Hence $f_i^{jk} = P_i^{jk}$, $g_{ki}^j = Q_{ki}^j$, and we recover formula (6.14). The uniqueness is proved. Consider now the element $\rho = e_i\varepsilon^i = (1/2)(e_i \otimes \varepsilon^i + \varepsilon^i \otimes e_i)$. Its coboundary in the Lie algebra cochain complex is the function $d\rho(u) = (-1)^{\tilde{u}}(S^2 \operatorname{ad} u)(\rho)$. Hence,

$$d\rho(e_k) = (-1)^{\tilde{k}}\Big([e_k, e_i]\varepsilon^i + (-1)^{\tilde{k}\tilde{\imath}}\varepsilon^i[e_k, e_i]\Big) = \tag{6.25}$$

$$(-1)^{\tilde{k}}\Big((-1)^{\tilde{\imath}} Q_{ki}^l e_l \varepsilon^i + (-1)^{\tilde{k}\tilde{\imath}}\varepsilon^i (P_k^{il} e_l + \varepsilon^l Q_{lk}^i)\Big) = (-1)^{\tilde{k}} P_k^{il} e_l e_i,$$

where we omitted a straightforward simplification. Similarly,

$$d\rho(\varepsilon^k) = (-1)^{\tilde{k}+1}\left([\varepsilon^k, e_i]\varepsilon^i + (-1)^{(\tilde{k}+1)\tilde{\imath}}\varepsilon^i[\varepsilon^k, e_i]\right) = \tag{6.26}$$

$$(-1)^{\tilde{k}+1}\left((-1)^{\tilde{k}}\varepsilon^i[e_i, \varepsilon^k] + (-1)^{(\tilde{k}+1)\tilde{\imath}}\varepsilon^i[\varepsilon^k, e_i]\right) =$$

$$(-1)^{\tilde{k}+1}\left((-1)^{\tilde{k}}\varepsilon^i(P_i^{kl}e_l + \varepsilon^l Q_{li}^k) + (-1)^{(\tilde{k}+1)\tilde{\imath}}\varepsilon^i(-1)^{\tilde{k}+1}\varepsilon^l P_l^{ki}\right) = -\varepsilon^i\varepsilon^l Q_{li}^k.$$

Thus, $d\rho = \delta$. □

THEOREM 6.2. *Let the vector space $\mathfrak{d} = \mathfrak{a} \oplus \mathfrak{b}$ have a structure of a Lie superalgebra with an invariant odd inner product. Suppose that the subspaces $\mathfrak{a}$ and $\mathfrak{b}$ are isotropic subalgebras in $\mathfrak{d}$. Then $\mathfrak{b} \cong \Pi\mathfrak{a}^*$, $\mathfrak{a} \cong \Pi\mathfrak{b}^*$, and $\mathfrak{a}$, $\mathfrak{b}$ are odd Lie bialgebras which are in odd duality. The Lie superalgebra structure on $\mathfrak{d}$ is isomorphic to that of the odd double either of $\mathfrak{a}$ or of $\mathfrak{b}$.*

PROOF. It is possible to pick bases $e_i \in \mathfrak{a}$ and $\varepsilon^i \in \mathfrak{b}$, $\tilde{e}_i = \tilde{\imath}$, $\tilde{\varepsilon}^i = \tilde{e}_i + 1 = \tilde{\imath} + 1$ such that $(e_i, \varepsilon^j) = \delta_i{}^j$, $(\varepsilon^j, e_i) = -\delta_i{}^j$. By an argument as in the proof of the previous theorem, we immediately recover the commutators of the basis elements in the form (6.13–6.15). Because we are given that $\mathfrak{d}$ is a Lie superalgebra, the vector field $\hat{Q}_D$ defining the brackets (6.13–6.15) is homological. Hence, its coefficients obey the relations that can be written as $\{Q, Q\} = 0$, $\{P, P\} = 0$, $\{Q, P\} = 0$, and we arrive at an odd bialgebra structure in $\mathfrak{a}$. The rest is obvious. Notice that $\mathfrak{a}$ and $\mathfrak{b}$ can be interchanged. □

In the classical case of Drinfeld's Lie bialgebras, one of the immediate applications of Drinfeld's double and of the concept of Manin's triples is a natural bialgebra structure in the classical Lie algebras like $\mathfrak{gl}(n)$ or $\mathfrak{sl}(n)$ and, more generally, in symmetrizable Kac–Moody Lie algebras. Roughly speaking, these algebras turn out to be "almost" doubles, i.e, are obtained from doubles by the factorization by a Lie bialgebra ideal isomorphic to an "extra copy" of a Cartan subalgebra (see [**3**] for details). As it turns out, this does not carry over straightforwardly to obvious candidates for an odd Lie bialgebra such as the Lie superalgebra $\mathfrak{q}(n)$ (see below). However, we can succeed by slightly generalizing the notion of the double, as follows.

Consider a Lie (super)algebra $\mathfrak{a}$ and a Lie (super)algebra $\mathfrak{h}$. We suppose the following. Let $\mathfrak{h}$ act on $\mathfrak{a}$ by derivations, and let $\mathfrak{h}$ be endowed with an invariant inner product of parity α. For $\alpha = 0$ or $\alpha = 1$ assume that there is a Lie bracket in $\mathfrak{a}^*$ or $\Pi\mathfrak{a}^*$, respectively. Let the contragredient action of $\mathfrak{h}$ on $\mathfrak{a}^*$ or $\Pi\mathfrak{a}^*$ be also by derivations. Denote $\mathfrak{b} := \mathfrak{a}^*$ or $\mathfrak{b} := \Pi\mathfrak{a}^*$ as appropriate. Then $\mathfrak{a} \oplus \mathfrak{h} \oplus \mathfrak{b}$ will be a potential "double of $\mathfrak{a}$ (or $\mathfrak{a} \oplus \mathfrak{h}$) over $\mathfrak{h}$". More precisely, this vector space has a natural inner product of parity α. Require a Lie bracket on $\mathfrak{a} \oplus \mathfrak{h} \oplus \mathfrak{b}$ such that both $\mathfrak{a} \oplus \mathfrak{h}$ and $\mathfrak{b} \oplus \mathfrak{h}$ (with natural brackets) are subalgebras and the inner product is invariant. It is a straightforward check that such conditions specify the bracket uniquely (if it exists). This is completely similar, e.g., to proofs of Drinfeld's theorem or Theorem 6.1. The resulting formula for the "cross" bracket would be

$$[e_i, e^j] = \pm Q_i^{jk}e_k \pm e^k Q_{ki}^j \pm Q_{i\mu}^j g^{\mu\lambda}e_\lambda; \tag{6.27}$$

here we use the notation e^j for the dual basis either in $\mathfrak{a}^*$ or in $\Pi\mathfrak{a}^*$, and $[e_i, e_j] = \pm Q_{ij}^k e_k$, $[e^i, e^j] = \pm e^k Q_k^{ij}$, $[e_\mu, e_i] = \pm Q_{i\mu}^k e_k$ (the last formula stands for the action of $\mathfrak{h}$ on $\mathfrak{a}$), and $g^{\mu\lambda}$ is the inverse Gram matrix for $\mathfrak{h}$. We call $\mathfrak{a}$ a *Lie bialgebra over*

$\mathfrak{h}$ (*of parity* α) if in $\mathfrak{a}\oplus\mathfrak{h}\oplus\mathfrak{b}$ is obtained a genuine Lie bracket. Consider an invariant element $e_i e^i$ of parity α, where the product means either the wedge product or the symmetric product. Applying ad of elements of $\mathfrak{a}\oplus\mathfrak{h}\oplus\mathfrak{b}$, we arrive at a cobracket δ on $\mathfrak{a}\oplus\mathfrak{h}\oplus\mathfrak{b}$ (even or odd). One can see that δ vanishes on $\mathfrak{h}$ (because $\mathfrak{h}$ acts, up to a sign, by the adjoint operators on e_i and e^i) and that

$$\delta(e_k) = \pm Q_k^{ij} e_i e_j \pm Q_{k\mu}^{j} g^{\mu\lambda} e_j e_\lambda, \tag{6.28}$$

$$\delta(e^k) = \pm Q_{ij}^{k} e^i e^j \pm Q_{j\mu}^{k} g^{\mu\lambda} e^j e_\lambda. \tag{6.29}$$

Immediately follows that cobrackets (6.28) and (6.29) separately satisfy co-Jacobi, because they are nothing but the cobrackets dual to the Lie brackets in $\mathfrak{b}\oplus\mathfrak{h}$ and $\mathfrak{a}\oplus\mathfrak{h}$, respectively. Hence, together we have a Lie bracket in the dual (or antidual, depending on α) space to $\mathfrak{g} = \mathfrak{a}\oplus\mathfrak{h}\oplus\mathfrak{b}$. By the construction, it is an ad $\mathfrak{g}$-cocycle. Hence $\mathfrak{g} = \mathfrak{a}\oplus\mathfrak{h}\oplus\mathfrak{b}$ is a Lie bialgebra (even or odd, depending on α), which is, by definition, the *double of* $\mathfrak{a}\oplus\mathfrak{h}$ *over* $\mathfrak{h}$. This argument can be reversed, leading to such a "relative double" structure in a Lie (super)algebra with an invariant inner product and an appropriate decomposition into $\mathfrak{a}\oplus\mathfrak{h}\oplus\mathfrak{b}$. In particular, the even case examples with Kac–Moody algebras (see [**3**]) are covered by this construction. For us the main motivation is the following example (where we get an odd bialgebra).

EXAMPLE 6.2. Consider the matrix Lie superalgebra $\mathfrak{q}(n)$. Recall that

$$\mathfrak{q}(n) = \big\{x \in \mathfrak{gl}(n|n) \,\big|\, [x, I] = 0\big\}, \text{ where } I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.30}$$

It consists of even matrices of the appearance

$$x = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{6.31}$$

and odd matrices of the appearance

$$x = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}. \tag{6.32}$$

(For the numerical entries, $b = 0$ in (6.31) and $a = 0$ in (6.32).) There is an invariant odd inner product on $\mathfrak{q}(n)$, defined as

$$(x, y) = (-1)^{\tilde{x}} \operatorname{otr}(xy), \tag{6.33}$$

where the "odd trace" is an (essentially unique) trace on $\mathfrak{q}(n)$ given by the formula

$$\operatorname{otr} x = \frac{1}{2}\operatorname{str}(Ix) = \operatorname{tr} b. \tag{6.34}$$

There is an analog of the Cartan decomposition: $\mathfrak{q}(n) = \mathfrak{n}_+\oplus\mathfrak{h}\oplus\mathfrak{n}_-$ into the "upper-triangular", "lower-triangular" and "diagonal" matrices. The Cartan subalgebra $\mathfrak{h} \subset \mathfrak{q}(n)$ is no longer commutative. It consists of n copies of the $1|1$-dimensional superalgebra considered in Example 6.1. (It is exactly the non-commutativity of $\mathfrak{h}$ that prevents to define a bialgebra structure in $\mathfrak{q}(n)$ similarly to $\mathfrak{gl}(n)$.) We can apply the above considerations and interpret $\mathfrak{q}(n)$ as a "relative" double of $\mathfrak{n}_+\oplus\mathfrak{h}$ over $\mathfrak{h}$. Thus $\mathfrak{q}(n)$ has a natural structure of an odd bialgebra. Direct calculations of commutators yield the following explicit formulas for the odd cobracket

$$\delta\colon \mathfrak{q}(n) \to S^2\mathfrak{q}(n). \tag{6.35}$$

For $k = l$:

$$\delta(e_{kk}) = 0, \quad \delta(\varepsilon_{kk}) = 0; \tag{6.36}$$

for $k < l$:

$$\delta(e_{kl}) = 2 \sum_{k<i<l} (-e_{ki}\varepsilon_{il} + \varepsilon_{ki}e_{il}) + (\varepsilon_{kk} - \varepsilon_{ll})e_{kl} - (e_{kk} - e_{ll})\varepsilon_{kl}, \tag{6.37}$$

$$\delta(\varepsilon_{kl}) = 2 \sum_{k<i<l} (e_{ki}e_{il} - \varepsilon_{ki}e_{il}) + (e_{kk} + e_{ll})e_{kl} - (\varepsilon_{kk} - \varepsilon_{ll})\varepsilon_{kl}; \tag{6.38}$$

for $k > l$:

$$\delta(e_{kl}) = 2 \sum_{l<i<k} (e_{ki}\varepsilon_{il} - \varepsilon_{ki}e_{il}) - (\varepsilon_{kk} - \varepsilon_{ll})e_{kl} + (e_{kk} - e_{ll})\varepsilon_{kl}, \tag{6.39}$$

$$\delta(\varepsilon_{kl}) = 2 \sum_{l<i<k} (-e_{ki}e_{il} + \varepsilon_{ki}e_{il}) - (e_{kk} + e_{ll})e_{kl} + (\varepsilon_{kk} - \varepsilon_{ll})\varepsilon_{kl}. \tag{6.40}$$

Here we use the basis $e_{ij}, \varepsilon_{ij} \in \mathfrak{q}(n)$, where $e_{ij} = \operatorname{diag}(E_{ij}, E_{ij}) \in \mathfrak{q}(n)_0$, $\varepsilon_{ij} = \operatorname{antidiag}(E_{ij}, E_{ij}) \in \mathfrak{q}(n)_1$, and E_{ij} are the usual matrix units. Note that $(e_{ij}, \varepsilon_{ji}) = -(\varepsilon_{ij}, e_{ji}) = 1$ and all other inner products of basis vectors are zero. A basis in $\mathfrak{n}_+$ is e_{ij}, ε_{ij} with $i < j$, a basis in $\mathfrak{n}_-$ is e_{ij}, ε_{ij} with $i > j$, and a basis in $\mathfrak{h}$ is e_{ii}, ε_{ii}.

(Notice that for $\mathfrak{q}(1)$ formulas (6.36) give zero cobracket, so the odd bialgebra structure given in Example 6.1 is different from the above.)

7. Appendix

7.1. Cotangent bundles for dual vector bundles. Here we give a full proof of Theorem 3.2. The following statement, for purely even vector bundles, appeared for the first time, to my knowledge, in the paper by Mackenzie and Xu [**17**]. (Earlier Tulczyjew [**25**] considered the particular case of tangent bundles.)

THEOREM 7.1. *For an arbitrary vector bundle $E \to M$ there is a natural diffeomorphism of the cotangent bundles $F\colon T^*E \cong T^*E^*$ which preserves the symplectic structure. It is a natural transformation in the categorical sense, and $F^2\colon T^*E \to T^*E^{**} \cong T^*E$ is the transformation induced by the multiplication by -1 in the fibers of E. (The diffeomorphism F is given by the explicit formula* (7.6) *below.)*

PROOF. Let us denote by x^a local coordinates on M, by y^i coordinates in the fiber of $E \to M$, by y_i the corresponding coordinates in the fiber of the dual bundle $E^* \to M$. Then a change of coordinates in E has the form

$$\begin{cases} x^a = x^a(x') \\ y^i = y^{i'} T_{i'}{}^i(x') \end{cases} \qquad \begin{cases} x^a = x^a(x') \\ y_i = T_i{}^{i'}(x')\, y_{i'} \end{cases} \tag{7.1}$$

(Here $T_i{}^{i'} T_{i'}{}^j = \delta_i{}^j$.) Denote the conjugate momenta in T^*E and T^*E^* by p_a, p_i and p_a, p^i, respectively. For T^*E we obtain

$$\begin{cases} p_a = \dfrac{\partial x^{a'}}{\partial x^a} p_{a'} + \dfrac{\partial y^{i'}}{\partial x^a} p_{i'} = \dfrac{\partial x^{a'}}{\partial x^a} p_{a'} + (-1)^{\tilde{a}\tilde{i}} y^{k'} T_{k'}{}^i \dfrac{\partial T_i{}^{i'}}{\partial x^a} p_{i'} \\ p_i = \dfrac{\partial y^{i'}}{\partial y^i} p_{i'} = T_i{}^{i'}(x') p_{i'} \end{cases} \tag{7.2}$$

and for T^*E^* we obtain

$$\text{(7.3)}\qquad \begin{cases} p_a = \dfrac{\partial x^{a'}}{\partial x^a} p_{a'} + \dfrac{\partial y_{k'}}{\partial x^a} p^{k'} = \dfrac{\partial x^{a'}}{\partial x^a} p_{a'} + \dfrac{\partial T_{k'}{}^i}{\partial x^a} T_i{}^{i'} y_{i'} p^{k'} \\ p^i = \dfrac{\partial y_{i'}}{\partial y_i} p^{i'} = (-1)^{\tilde{\imath}(\tilde{\imath}+\tilde{\imath}')} T_{i'}{}^i(x') p^{i'}. \end{cases}$$

(We assume that $\tilde{x}^a = \tilde{a}$, $\tilde{y}^i = \tilde{\imath}$.) Notice that $(-1)^{\tilde{\imath}(\tilde{\imath}+\tilde{\imath}')} T_{i'}{}^i p^{i'} = (-1)^{\tilde{\imath}+\tilde{\imath}'} p^{i'} T_{i'}{}^i$ and

$$\text{(7.4)}\qquad \frac{\partial T_{k'}{}^i}{\partial x^a} T_i{}^{i'} + (-1)^{\tilde{a}(\tilde{k}'+\tilde{\imath})} T_{k'}{}^i \frac{\partial T_i{}^{i'}}{\partial x^a} = 0,$$

hence

$$\frac{\partial T_{k'}{}^i}{\partial x^a} T_i{}^{i'} y_{i'} p^{k'} = -(-1)^{\tilde{a}(\tilde{k}'+\tilde{\imath})+\tilde{k}'(\tilde{k}'+\tilde{a})} p^{k'} T_{k'}{}^i \frac{\partial T_i{}^{i'}}{\partial x^a} y_{i'} = -(-1)^{\tilde{a}\tilde{\imath}+\tilde{k}'} p^{k'} T_{k'}{}^i \frac{\partial T_i{}^{i'}}{\partial x^a} y_{i'}.$$

We see that it is possible to rewrite the change of coordinates on T^*E^* as

$$\text{(7.5)}\qquad \begin{cases} x^a = x^a(x') \\ y_i = T_i{}^{i'}(x')\, y_{i'} \\ p_a = \dfrac{\partial x^{a'}}{\partial x^a} p_{a'} - (-1)^{\tilde{a}\tilde{\imath}+\tilde{k}'} p^{k'} T_{k'}{}^i \dfrac{\partial T_i{}^{i'}}{\partial x^a} y_{i'} \\ (-1)^{\tilde{\imath}} p^i = (-1)^{\tilde{\imath}'} p^{i'} T_{i'}{}^i \end{cases}$$

which is identical with that for T^*E if we substitute $y_i := p_i$, $p^i := (-1)^{\tilde{\imath}+1} y^i$. Hence, we define the desired diffeomorphism $F\colon T^*E \to T^*E^*$ as

$$\text{(7.6)}\qquad (x^a, y^i, p_a, p_i) \mapsto (x^a, y_i, p_a, p^i) = (x^a, p_i, p_a, -(-1)^{\tilde{\imath}} y^i).$$

Consider the symplectic form $dp_a dx^a + dp^i dy_i$ on T^*E^*. Substituting (7.6), we get $dp_a dx^a + (-1)^{\tilde{\imath}+1} dy^i dp_i = dp_a dx^a + dp_i dy^i$, which is the symplectic form on T^*E. Thus, the diffeomorphism (7.6) is a canonical transformation. Notice, finally, that formulas (7.6) define the transformation F for all vector bundles. One only has to be careful with the distinction of left and right coordinates: our formulas make use of left coordinates in E and the corresponding right coordinates in E^*. (If we want to change, say, a left coordinate y^i into the right coordinate, we have to multiply both y^i and the corresponding momentum by $(-1)^{\tilde{\imath}}$.) Rewritten in left coordinates only, the formula for F becomes

$$\text{(7.7)}\qquad F_E\colon\ (x^a, y^i, p_a, p_i) \mapsto (x^a, \bar{y}_i, p_a, \bar{p}^i) = (x^a, (-1)^{\tilde{\imath}} p_i, p_a, -y^i)$$

where bar is used to denote the left coordinates in E^* and the respective momenta. (The natural pairing of E and E^* is $y^i y_i = (-1)^{\tilde{\imath}} y^i \bar{y}_i$.) Hence, for F_{E^*} we will get

$$\text{(7.8)}\qquad F_{E^*}\colon\ (x^a, \bar{y}_i, p_a, \bar{p}^i) \mapsto (x^a, w^i, p_a, q_i) = (x^a, (-1)^{\tilde{\imath}} \bar{p}^i, p_a, -\bar{y}_i).$$

At the right hand side w^i stand for (left) coordinates in E^{**} and q_i for the corresponding momenta. Notice that the natural isomorphism between E^{**} and E is, in left coordinates, $I_E\colon\ (x^a, w^i) \mapsto (x^a, y^i) = (x^a, (-1)^{\tilde{\imath}} w^i)$. Hence

$$\text{(7.9)}\qquad I_E \circ F_{E^*}\colon\ (x^a, \bar{y}_i, p_a, \bar{p}^i) \mapsto (x^a, y^i, p_a, p_i) = (x^a, \bar{p}^i, p_a, -(-1)^{\tilde{\imath}} \bar{y}_i).$$

Taking composition with F_E and using formula (7.7) we finally obtain:

$$\text{(7.10)}\qquad I_E \circ F_{E^*} \circ F_E\colon\ (x^a, y^i, p_a, p_i) \mapsto (x^a, y^i, p_a, p_i) = (x^a, -y^i, p_a, -p_i),$$

which concludes the proof, if we identify E^{**} with E with the help of I_E. □

7.2. Analog for anticotangent bundles.

THEOREM 7.2. *For an arbitrary vector bundle $E \to M$ there is a natural diffeomorphism (in the categorical sense) of the anticotangent bundles*

$$F\colon\ \Pi T^*E \cong \Pi T^*(\Pi E^*), \tag{7.11}$$

which preserves the canonical odd symplectic structure. It is given by formula (7.16) *below. The square $F^2\colon\ \Pi T^*E \to \Pi T^*(\Pi(\Pi E^*)^*) \cong \Pi T^*E$ is the transformation induced by the multiplication by -1 in the fibers of E.*

PROOF. Let coordinates in E be denoted, as above, by x^a and y^i. Let x^*_a and y^*_i be the corresponding odd momenta. Then, similarly to (7.2), we have

$$\begin{cases} x^a = x^a(x') \\ y^i = y^{i'} T_{i'}{}^i(x') \\ x^*_a = \dfrac{\partial x^{a'}}{\partial x^a} x^*_{a'} + (-1)^{\tilde{a}\tilde{i}} y^{k'} T_{k'}{}^i \dfrac{\partial T_i{}^{i'}}{\partial x^a} y^*_{i'} \\ y^*_i = T_i{}^{i'}(x') x^*_{i'} \end{cases} \tag{7.12}$$

Consider now the antidual bundle ΠE^*. Let η_i denote the fiber coordinates in ΠE^* that are right-contragredient to y^i, i.e., the form $y^i\eta_i$ gives the invariant canonical odd pairing of E and ΠE^*. Notice that $\tilde{\eta}_i = \tilde{y}^i + 1 = \tilde{\imath} + 1$. Then

$$\eta_i = T_i{}^{i'}(x')\eta_{i'} = (-1)^{(\tilde{\imath}'+1)\tilde{\imath}} \eta_{i'} T_i{}^{i'}(x'), \tag{7.13}$$

and we get

$$\begin{cases} x^a = x^a(x') \\ \eta_i = T_i{}^{i'}(x')\eta_{i'} \\ x^*_a = \dfrac{\partial x^{a'}}{\partial x^a} x^*_{a'} + \dfrac{\partial T_{k'}{}^i}{\partial x^a} T_i{}^{i'} \eta_{i'} \eta^{*k'} \\ \eta^{*i} = (-1)^{(\tilde{\imath}+1)\tilde{\imath}'} T_{i'}{}^i(x') \eta^{*i'} \end{cases} \tag{7.14}$$

for the changes of coordinates in $\Pi T^*(\Pi E^*)$. Notice that $\tilde{\eta}^{*i} = \tilde{\eta}_i + 1 = \tilde{\imath}$. Using formula (7.4) and moving $\eta^{*k'}$ to the left, we can rewrite it as

$$\begin{cases} x^a = x^a(x') \\ \eta_i = T_i{}^{i'}(x')\eta_{i'} \\ x^*_a = \dfrac{\partial x^{a'}}{\partial x^a} x^*_{a'} - (-1)^{\tilde{a}\tilde{i}} \eta^{*k'} T_{k'}{}^i \dfrac{\partial T_i{}^{i'}}{\partial x^a} \eta_{i'} \\ \eta^{*i} = \eta^{*i'} T_{i'}{}^i(x') \end{cases} \tag{7.15}$$

Hence, we can define the desired diffeomorphism between ΠT^*E and $\Pi T^*(\Pi E^*)$ by the formula

$$F\colon\ (x^a, y^i, x^*_a, y^*_i) \mapsto (x^a, \eta_i, x^*_a, \eta^{*i}) = (x^a, y^*_i, x^*_a, -y^i). \tag{7.16}$$

Check the symplectic form: on $\Pi T^*(\Pi E^*)$ it is $\omega = (-1)^{\tilde{a}+1} dx^*_a dx^a + (-1)^{\tilde{\imath}} d\eta^{*i} d\eta_i$, and substituting $\eta_i := y^*_i$, $\eta^{*i} := -y^i$ we get $(-1)^{\tilde{a}+1} dx^*_a dx^a - (-1)^{\tilde{\imath}} dy^i dy^*_i = (-1)^{\tilde{a}+1} dx^*_a dx^a + (-1)^{\tilde{\imath}+1} dy^*_i dy^i$, which is exactly the odd symplectic form on ΠT^*E. We omit the direct check for F^2, which is completely similar to that in the proof of Theorem 7.1. □

EXAMPLE 7.1. Let $\mathfrak{g}$ be a vector space. Consider an even linear bivector field $P = (1/2)\xi^k P_k^{ij} \xi_j^* \xi_i^*$ on $\Pi\mathfrak{g}$. According to our theorem, there is a diffeomorphism $\Pi T^*\Pi\mathfrak{g} \to \Pi T^*\mathfrak{g}^*$ (preserving the odd symplectic structure), which in coordinates is given by $x_i = \xi_i^*$, $x^{*i} = -\xi^i$. Hence, we can transform P into the function $-(1/2)x^{*k}P_k^{ij}x_j x_i = -(1/2)(-1)^{\tilde{k}+1}P_k^{ij}x_j x_i x^{*k}$ on $\Pi T^*\mathfrak{g}^*$, which is an even antivector field on $\mathfrak{g}^*$. Recalling θ: $\mathrm{Vect}(\mathfrak{g}^*) \to C^\infty(\Pi T^*\mathfrak{g}^*)$, $X = X_k(x)\partial/\partial x_k \mapsto (-1)^{\tilde{X}}X_k(x)x^{*k}$, we recover an odd vector field $\hat{Q}^* = (1/2)(-1)^{\tilde{k}+1}P_k^{ij}x_j x_i \partial/\partial x_k$. Because the symplectic structure is preserved, the field $\hat{Q}^*$ on $\mathfrak{g}^*$ is homological if and only if P is a Poisson tensor.

PROPOSITION 7.1. *The Lie bracket in $\Pi\mathfrak{g}^*$ defined by the field $\hat{Q}^*$ is given by the formulas*

$$[\varepsilon^i, \varepsilon^j] = (-1)^{\tilde{\imath}+1}\varepsilon^k P_k^{ij}. \tag{7.17}$$

Hence, the corresponding Lie–Poisson bracket on $\Pi\mathfrak{g}$ coincides with the Poisson bracket directly given by the tensor P.

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Department of Mathematics, University of Manchester Institute of Science and Technology (UMIST), PO Box 88, Manchester M60 1QD, England
E-mail address: `theodore.voronov@umist.ac.uk`

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On the structure of graded symplectic supermanifolds and Courant algebroids.

Dmitry Roytenberg

ABSTRACT. This paper is devoted to a study of geometric structures expressible in terms of graded symplectic supermanifolds. We extend the classical BRST formalism to arbitrary pseudo-Euclidean vector bundles $E \to M_0$ by canonically associating to such a bundle a graded symplectic supermanifold (M, Ω), with $\deg(\Omega) = 2$. Conversely, every such manifold arises in this way. We describe the algebra of functions on M in terms of E and show that "BRST charges" on M correspond to Courant algebroid structures on E, thereby constructing the standard complex for the latter as a generalization of the classical BRST complex. As an application of these ideas, we prove the acyclicity of "higher de Rham complexes", a generalization of a classic result of Fröhlicher-Nijenhuis, and derive several easy but useful corollaries.

1. Introduction.

Graded symplectic supermanifolds have been known to physicists since the 70s, providing framework for the so-called BRST formalism. Such supermanifolds were obtained by adjoining odd generators ("ghosts" and "antighosts") to the classical algebra of observables, and having them satisfy canonical Poisson bracket relations. In the simplest case, such supermanifolds are of the form $M = \Pi(\mathfrak{g} \oplus \mathfrak{g}^*) \times T^*M_0$, where M_0 is the configuration space acted upon by a Lie algebra $\mathfrak{g}$, and Π denotes parity shift. The symplectic form is $\Omega = dp_i dq^i + d\theta_a d\xi^a$, where the coordinates (ξ^a) on $\Pi\mathfrak{g}$ are the "ghosts", those (θ_a) on $\Pi\mathfrak{g}^*$ – the "antighosts". One assigns the ghost degree $+1$ to the ghost variables, while the antighosts are assigned degree -1. The symplectic form Ω has total ghost degree zero, so that one can replace T^*M_0 by an arbitrary symplectic manifold P acted upon by $\mathfrak{g}$. The classical BRST algebra is the graded Poisson algebra $\wedge^\cdot(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes C^\infty(P)$ of functions on M. One then constructs an odd self-commuting Hamiltonian of ghost degree $+1$ (the "BRST charge") which acts on the BRST algebra via the Poisson bracket as a differential; its cohomology (the BRST cohomology) is identified with the "physical" observables.

A detailed mathematical study of the above special case of classical BRST algebra and its quantization was undertaken by Kostant and Sternberg [7] who related the BRST cohomology with Lie algebra cohomology. They showed, in particular,

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that if the action of $\mathfrak{g}$ on P is "nice", the BRST cohomology is concentrated in degree 0 and is isomorphic to the algebra of functions on the reduced phase space.

The classical BRST formalism can be extended and generalized in various directions. For instance, to treat more complicated symmetries it is required to further extend the BRST algebra by including "ghosts for ghosts" etc. The homological algebra becomes quite involved in this case (see [**4**]). The purpose of this paper is to generalize the BRST formalism to the case of an arbitrary pseudo-Euclidean vector bundle E over a manifold M_0, and to study the resulting geometric structures, Courant algebroids. Consider the special case $M_0 = \text{point}$, treated in [**7**]. Here we have a pseudo-Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$, $M = \Pi V$, $\Omega = \frac{1}{2} d\xi^a g_{ab} d\xi^b$, where $g_{ab} = \langle e_a, e_b \rangle$ for some basis (e_a) of V. Not having a ghost-antighost splitting, we assign degree $+1$ to all ξ^a's, so that the degree of Ω is now $+2$. The exterior algebra $\wedge^\cdot V^*$ becomes a graded Poisson algebra; the Poisson bracket has degree -2. Hence, the space $\wedge^2 V^*$ of quadratic Hamiltonians becomes a Lie algebra, isomorphic to $\mathfrak{so}(V)$. It acts on all Grassman polynomials $\wedge^k V^*$ via the Poisson bracket. The situation is entirely analogous to the well-known realization of the Lie algebra $\mathfrak{sp}(V)$ of a symplectic vector space (V, ω) as the Lie algebra of quadratic Hamiltonians $S^2 V^*$ under the canonical Poisson bracket.

If V carries a Lie algebra structure such that $\langle \cdot, \cdot \rangle$ is ad-invariant, one forms the *Cartan structure tensor* $\Theta \in \wedge^3 V^*$ by setting $\Theta(X, Y, Z) = \langle [X, Y], Z \rangle$. It plays the role of a "BRST charge" for the standard complex of the Lie algebra V: it satisfies the structure equation $\{\Theta, \Theta\} = 0$ (equivalent to the Jacobi identity), and $\delta = \{\Theta, \cdot\}$ is the Chevalley-Eilenberg differential. The Lie bracket on V is recovered as a so-called *derived bracket*: $[X, Y] = \{\{X, \Theta\}, Y\}$ (we have used $\langle \cdot, \cdot \rangle$ to identify V with V^*). This formalism was exploited by Lecomte–Roger [**8**] and Kosmann-Schwarzbach [**6**] who used it to study the homological algebra of Lie bialgebras and quasi-Lie bialgebras, respectively.

If $M_0 \times V$ is a trivial pseudo-Euclidean vector bundle over M_0, one sets $M = \Pi V \times T^* M_0$, with $\Omega = dp_i dq^i + \frac{1}{2} d\xi^a g_{ab} d\xi^b$. One needs to assign degree 2 to the momenta p_i so that Ω is homogeneous of degree 2. Thus it is proper to denote $M = V[1] \times T^*[2] M_0$, to indicate the shift in grading. Note that with this choice of grading every homogeneous function on M has a nonnegative degree compatible with its parity: M is a so-called *N-manifold* [**19**]. It turns out that one can canonically associate a symplectic N-manifold M of degree 2 to any, not necessarily trivial, pseudo-Euclidean vector bundle $E \to M_0$, and that conversely, every such manifold arisies in this way (Theorem 3.3). Local trivializations of E (as a pseudo-Euclidean bundle) give rise to affine Darboux charts of the form $V[1] \times T^*[2] U$, and M is glued together from these by affine coordinate transformations (3.3) corresponding to changes of trivialization of E. An observation, often overlooked, is that all canonical constructions in the category of vector bundles should be equivariant with respect to the group of bundle automorphisms covering diffeomorphisms of the base (as opposed to just those that leave every point of the base fixed). The Lie algebra of this group is known as the *Atiyah algebra*, consisting of so-called covariant differential operators (CDO's) on E. In case of pseudo-Euclidean vector bundles, one restricts to the subalgebra of operators preserving $\langle \cdot, \cdot \rangle$. This algebra is realized as the Lie algebra $\mathcal{A}^2$ of quadratic Hamiltonians on M, under the Poisson bracket. It acts on all polynomial functions on M, preserving the degree.

Whereas in studying structures on vector spaces (such as Lie algebras, bialgebras, etc.) the use of supermanifold formalism is entirely optional and a matter of one's taste, it offers significant advantages already when we go to vector bundles. The analogue of a Lie algebra in the world of vector bundles is a *Lie algebroid*. A Lie algebroid structure on a vector bundle $A \to M_0$ is usually described in terms of an anchor map $A \to TM_0$ and a Lie bracket of sections, but it can be equivalently described as a derivation d_A of $\Gamma(\wedge^\cdot A^*)$ of degree $+1$ and square zero. This differential can be viewed as a homological vector field on the N-manifold $A[1] = \Pi A$ (in the terminology of [**19**], an *NQ-structure*). The standard example is the tangent bundle TM_0 with identity anchor and the commutator bracket of vector fields; the complex in this case is just the de Rham complex of M_0. This homological description facilitates an elegant and concise formulation of such important notions as modules and homomoprhisms of Lie algebroids. The classical definition of homomorphism is rather complicated, especially when different base manifolds are involved; the classical notion of a module, while certainly reasonable, fails to include perhaps the most important module of all, the adjoint module (see [**10**] for a thorough discussion). The homological formulation overcomes these difficulties [**16**]. Its potential is yet to be fully realized.

As soon as a pseudo-Euclidean structure on a vector bundle $E \to M_0$ is introduced, one quickly realizes that any reasonable notion of "ad-invariance" is incompatible with the notion of Lie algebroid, except in the trivial case of a bundle of Lie algebras. One has to sacrifice the skew-symmetry of the bracket of sections by adding a symmetric part which is, in a suitable sense, infinitesimal (a coboundary), or else introduce anomalies into the Jacobi and other basic identities. Thus *Courant algebroids* are born. The first basic example (called *standard*), on $E = TM_0 \oplus T^*M_0$, is due to T. Courant [**2**], who used it to describe integrability of so-called *Dirac structures* on M_0 (including 2-forms, bivectors and distributions) in a unified way. The general definition appeared in [**9**] as a result of a "doubling" construction for *Lie bialgebroids* [**11**]. This leads, for a Lie bialgebroid (A, A^*), to a Courant algebroid structure on $E = A \oplus A^*$, generalizing Courant's example. Ševera [**18**] showed how Courant algebroids appear naturally in the context of two-dimensional variational problems. He also classified so-called *exact Courant algebroids*, deformations of the standard one by closed 3-forms. These were later used to describe twisted Poisson manifolds with a closed 3-form background ([**20**], see also [**15**]).

To describe Courant algebroids homologically, the (merely Poisson, not symplectic) N-manifold $E[1] = \Pi E$ is not enough; one needs its minimal symplectic realization (M, Ω) mentioned above. It turns out (Theorem 4.5) that Courant algebroid structures on E are in 1-1 correspondence with "BRST charges" on M, i.e. cubic Hamiltonians Θ satisfying the structure equation $\{\Theta, \Theta\} = 0$. The BRST differential $D = \{\Theta, \cdot\}$, acting on the algebra $\mathcal{A}^\cdot$ of polynomial functions on M, gives the standard complex for the Courant algebroid. For Lie bialgebroids this description (the "homological double") was given in [**14**] (in that case $M = T^*\Pi A$ with an appropriate assignment of grading). In case M_0 is a point, we recover Cartan's structure tensor. At this point we should remark that instead of N-manifolds we could consider general graded manifolds [**17**]; all our constructions extend to this case at the small extra cost of having to consider the even and odd symplectic forms separately, as the degree is no longer tied to parity. This way we would

obtain even and odd Courant superalgebroids over supermanifolds. We stick with N-manifolds for simplicity of presentation.

One remarkable feature of Courant algebroids, apparent in the homological formulation, is that they can be used as target spaces for a general class of 3-dimensional topological field theories. Special cases include the Chern–Simons gauge theory (when M_0 is a point) and Park's topological membrane theory [**12**] (when E is an exact Courant algebroid). This fits in the general Batalin–Vilkovisky framework described in [**1**]. The potential applications to the topology of 3-manifolds are beginning to be investigated; the program was outlined to us by Park [**13**], who also considers symplectic NQ-manifolds of higher degree.

When $E = TM_0 \oplus T^*M_0$ is the standard Courant algebroid, $M = T^*\Pi TM_0$, and $\Theta = \Theta_0$ is just the Hamiltonian corresponding to the de Rham vector field d on ΠTM_0. The standard (BRST) complex gets decomposed into a series of subcomplexes $(\mathcal{A}^{k,\cdot}, D)$ which we call the *higher de Rham complexes*. For $k = 0$ this is just the de Rham complex of M_0, whereas $\mathcal{A}^{1,\cdot}[1]$ can be identified with vector fields on ΠTM_0, with $D = [d, \cdot]$; in general, $\mathcal{A}^{k,l}$ consists of the symbols of differential operators on ΠTM_0 of order k and degree $l - k$; $(\mathcal{A}^{k,\cdot}, D)$ can be interpreted as the standard complex of the Lie algebroid TM_0 with coefficients in the kth exterior power of the adjoint module.

We prove (Proposition 5.1) that the higher de Rham complexes are acyclic for $k > 0$ by constructing a chain homotopy for D as a vector field ι on M of degree -1. This generalizes the classic result of Frölicher and Nijenhuis [**3**] on the structure of derivations of the ring of differential forms. Several corollaries, potentially of independent interest, follow immediately from this: Cor. 5.2 (the rigidity of the standard Lie algebroid structure on TM_0), Cor. 5.3 (any Lie algebroid structure on T^*M_0 compatible with the standard one on TM_0 comes from a Poisson structure on M_0) and Cor. 5.4 (the cohomology of the standard Courant algebroid is isomorphic to the de Rham cohomology of M_0). This last corollary implies, in particular, that Ševera's exact Courant algebroids are the only nontrivial deformations of the standard one.

This paper is organized as follows. In Section 2 we review the generalities on graded manifolds and N-manifolds, following [**17**] and [**19**]; in Section 3 we study the structure of symplectic N-manifolds of degree 1 and 2 in detail; Section 4 is devoted to the homological (BRST) description of Courant algebroids, their standard complexes and cohomology; finally, in Section 5 we concentrate on the special case of the standard Courant algebroid and its deformations.

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2. Graded manifolds and N-manifolds.

A *graded manifold* is a (super)manifold with an additional grading in the structure sheaf. Specifically, it is a manifold M that possesses a coordinate atlas (of so-called *affine charts*) in which each local coordinate is assigned a *degree* (or *weight*), and the coordinate transformations are required to preserve the total weight. In this note we shall require these weights to be non-negative integers, although it is not a necessary restriction, in principle. The grading can also be conveniently

described by means of the *Euler vector field*[1]

$$\epsilon = \sum w(x^i) x^i \frac{\partial}{\partial x^i},$$

where $w(x^i)$ denotes the weight of x^i. The weights are just the eigenvalues with respect to the action of ϵ: we say that a function f is *homogeneous of weight* k if $\epsilon \cdot f = kf$. We shall denote the space of all such functions by $\mathcal{A}^k$ and call the graded algebra $\mathcal{A}^\cdot = \bigoplus_{k \geq 0} \mathcal{A}^k$ the *algebra of polynomial functions* on M; the algebra of all smooth functions is a completion of $\mathcal{A}^\cdot$. The *degree of* M is by definition the highest weight of a local coordinate.

Any (super)manifold M becomes a graded manifold if we assign zero weight to all the coordinates (i.e. set $\epsilon = 0$). The total space M_1 of any vector bundle A over a base (super)manifold M_0 becomes a graded manifold if the base coordinates are assigned zero weight, while all the fibre coordinates are assigned the same weight n. The standard choice, of course, is $n = 1$. In fact, any manifold of degree 1 is easily seen to be a vector bundle.

In general, different local coordinates can have different weights, and the grading can be thought of as a generalization of a vector bundle structure; a comparison with physical units of measurement is also helpful. A general non-negatively graded manifold M has the following structure. Let $\mathcal{A}_k$ denote the (graded) subalgebra of $\mathcal{A}^\cdot$ locally generated by functions of weight $\leq k$. Then $\mathcal{A}^\cdot$ is filtered by the $\mathcal{A}_k$'s:

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_d = \mathcal{A}^\cdot,$$

where d is the degree of M. Clearly, $\mathcal{A}^k = \mathcal{A}_k / \mathcal{A}_{k+1}$. Correspondingly, we have a tower of fibrations

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_d = M$$

Here M_0 is a (super)manifold, M_1 is a vector bundle over M_0, and for $k \geq 1$, $M_k \leftarrow M_{k+1}$ is an affine fibration. Each M_k is itself a graded manifold (of degree k), and their Euler vector fields are related by the projections. M_0 also embeds into each M_k as the set of zeros of the Euler vector field; it corresponds to the ideal $\mathcal{A}^{\geq 1}$.

In particular, $\mathcal{A}_0 = \mathcal{A}^0 = C^\infty(M_0)$, and all $\mathcal{A}_k$'s and $\mathcal{A}^k$'s are modules over $\mathcal{A}_0$. It is best to think of these as (locally free) sheaves of $C^\infty(M_0)$-modules over M_0, as they can be restricted to arbitrary open subsets of M_0. Thus, $\mathcal{A}^1$ is the sheaf of sections of some vector bundle over M_0, while $\mathcal{A}_1 = S^\cdot \mathcal{A}^1$, etc. In what follows we shall frequently commit an abuse of notation by using $\mathcal{A}^k$ to refer to both the vector bundle and its sheaf of sections.

A (non-negatively integer) graded supermanifold M is an *N-manifold* if the integer grading is compatible with parity (the underlying $\mathbb{Z}_2$-grading in the structure sheaf). In plain English, this means that even coordinates have even weights, while odd ones have odd weights. The parity operator is related to the Euler vector field via $P = (-1)^\epsilon$. Thus we have an action of the multiplicative semigroup $(\mathbb{R}, \times)$ on M: -1 acts as P, 0 picks out $\mathcal{A}^0$, while $\lambda > 0$ acts as λ^ϵ. Then $M_0 = 0 \cdot M$ is an ordinary manifold, while $M_1 = \Pi A$ for some ordinary vector bundle over M_0 (i.e. $\mathcal{A}_1 = \Gamma(\wedge^\cdot A^*)$). Conversely, given a vector bundle $A \to M_0$, we can define the N-manifold $A[n]$ by assigning degree n to the fibre coordinates; it is equal to either

[1]To avoid pathologies one must require the Euler vector field to be complete. This means the affine charts are "cylindrical" in the direction of coordinates of non-zero weight, even or odd.

A or ΠA, depending on the parity of n. The *support* of an N-manifold M is the ordinary graded submanifold M_{even} corresponding to the ideal $\mathcal{A}^{\text{odd}}$.

REMARK 2.1. More generally, if N is a graded manifold and $E \to N$ is a graded vector bundle, we denote by $E[n]$ the graded manifold obtained by shifting the fibre degrees by n. In particular, TN and T^*N are graded vector bundles, where the Euler vector field acts as a Lie derivative. With respect to this induced grading, the "velocities" have the same degree as the corresponding coordinates on N, whereas the "momenta" have the opposite degrees. If N is an N-manifold, we may also have to shift the parity in the fibers so that $E[n]$ is again an n-manifold. For example, if $A \to M_0$ is an ordinary vector bundle, $T[1]A[1]$ is $\Pi T(\Pi A)$ with coordinates $(x^i, \xi^a, dx^i, d\xi^a)$ of weights $w(x^i) = 0$, $w(\xi^a) = 1$, $w(dx^i) = 1$, $w(d\xi^a) = 2$, and $T^*[2]A[1]$ is $T^*\Pi A$ with coordinates $(x^i, \xi^a, p_i, \theta_a)$ of weights $w(x^i) = 0$, $w(\xi^a) = 1$, $w(p_i) = 2$, $w(\theta_a) = 1$.

Graded manifolds form a category: the morphisms are required to be ϵ-equivariant (weight-preserving). N-manifolds form a full subcategory.

Given a graded manifold M, the Euler vector field ϵ acts on all canonical objects (tensors, jets, etc.) on M via the Lie derivative, whereby these objects also acquire weights. For example, an *NQ-manifold* is an N-manifold endowed with an integrable (homological) vector field Q of weight $+1$. That is, $[\epsilon, Q] = Q$ and $[Q, Q] = 2Q^2 = 0$. Such a Q is necessarily odd, so the integrability condition is nontrivial, and $(\mathcal{A}^{\cdot}, Q)$ becomes a cochain complex. The classic example of an NQ-manifold is the anti-tangent bundle $T[1]M_0 = \Pi TM_0$ of an ordinary manifold M_0; functions on $T[1]M_0$ are differential forms on M_0, and $Q = d$ is the exterior derivative. In fact, the assignment $M_0 \mapsto (T[1]M_0, d)$ is a functor that embeds ordinary manifolds into NQ-manifolds as a full subcategory. Thus, NQ-manifolds should be thought of as a natural enlargement of the category of smooth manifolds: much of the calculus carries over.

More generally, NQ-manifolds of degree 1 are the same as Lie algebroids over the base M_0. For this reason it is natural to call NQ-manifolds of degree n *n-algebroids*. Interesting results and conjectures concerning the integration of n-algebroids are contained in [**19**].

Other objects of interest are homogeneous symplectic and Poisson structures. For instance, a symplectic structure of degree n is a closed non-degenerate 2-form Ω such that $\mathcal{L}_\epsilon \Omega = n\Omega$. The corresponding Poisson tensor W has weight $-n$. We shall also consider symplectic NQ-manifolds for which we have the additional condition $\mathcal{L}_Q \Omega = 0$.

Let us conclude this section with the following simple observations, showing that a homogeneous symplectic form places severe restrictions on the structure of M:

LEMMA 2.2. *Let M be a graded manifold. Then*

(1) Any homogeneous symplectic form Ω on M of weight $n \geq 1$ is exact; in fact,

$$\Omega = d\left(\frac{1}{n} \iota_\epsilon \Omega\right)$$

(2) Any homogeneous vector field V of weight $m > -n$ preserving Ω is Hamiltonian. In fact,

$$\iota_V \Omega = \pm d\left(\frac{1}{m+n} \iota_V \iota_\epsilon \Omega\right)$$

PROOF. Both statements are easy consequences of Cartan's homotopy formula. □

EXAMPLE 2.3. Given a manifold M_0, consider $M = T^*[1]M_0 = \Pi T^* M_0$, with $\Omega = d\theta_i dx^i = d(\theta_i dx^i)$ of degree $+1$. Functions on M correspond to multivector fields on M_0, and the Poisson bracket of degree -1 given by Ω is the Schouten bracket. Vector fields on M of degree -1 correspond to 1-forms on M_0 via $\alpha \leftrightarrow \iota_\alpha$. Such a vector field preserves Ω if and only if α is closed; it is Hamiltonian if and only if α is exact. This example illustates that the second statement of the Lemma above generally fails if $m = -n$.

LEMMA 2.4. *If (M, Ω) is a non-negatively graded symplectic manifold, the degree of M cannot exceed $w(\Omega)$.*

PROOF. In an affine chart, $\Omega = \frac{1}{2} dx^i \Omega_{ij}(x) dx^j$, and the total weight of each term is $w(\Omega)$. If there were a coordinate of weight higher than $w(\Omega)$, it would have to be missing from the expression, contradicting the nondegeneracy of Ω. □

3. Degree 1 and 2.

Now let us analyze symplectic structures of degree 1 and 2 in detail. We shall restrict our attention to N-manifolds, although the general case can be handled at the small additional cost of having to consider the even and odd cases separately.

As a warm-up exercise, let us do the degree 1 case. So let (M, Ω) be a symplectic N-manifold with $w(\Omega) = 1$. According to Lemma 2.4, the degree of M is at most 1, so $M = M_1 \to M_0$ is a vector bundle. More precisely, $M = M_1 = A^*[1] = \Pi A^*$ for some vector bundle A over a manifold M_0. It will be convenient to use the Poisson bracket defined by Ω. It is odd, nondegenerate and has degree -1. We have $\{\mathcal{A}^0, \mathcal{A}^0\} = 0$, $\{\mathcal{A}^1, \mathcal{A}^0\} \subset \mathcal{A}^0$ and $\{\mathcal{A}^1, \mathcal{A}^1\} \subset \mathcal{A}^1$, hence the structure is determined by the Hamiltonian action of a Lie algebra $\mathcal{A}^1 = \Gamma(A)$ on $\mathcal{A}^0 = C^\infty(M_0)$. According to the Leibniz rule, this action is given by a vector bundle map $A \to TM_0$, which is an isomorphism by the nondegeneracy of $\{\cdot, \cdot\}$. Finally, by Jacobi, the Poisson bracket on $\mathcal{A}^1$ corresponds under this isomorphism to the commutator of vector fields on M_0. Thus, we have shown that symplectic N-manifolds of degree 1 are exhausted by Example 2.3:

PROPOSITION 3.1. *N-manifolds with a symplectic structure of degree 1 are in one-to-one correspondence with ordinary smooth manifolds, via $N \leftrightarrow (T^*[1]N, \Omega)$, where Ω is determined by the Schouten bracket of multivector fields.*

Let us notice two things. First, the Lie algebra of symmetries of (M, Ω) (Hamiltonian by Lemma 2.2) is $\mathcal{A}^1$, which is identified with the Lie algebra of vector fields on M_0. Second, a local coordinate chart $\{x^i\}$ on M_0 gives rise automatically to an affine Darboux chart $\{x^i, \theta_i\}$ on M, in which $\Omega = d\theta_i dx^i = d(\theta_i dx^i)$.

Now let us tackle the degree 2 case. By Lemma 2.4, such a manifold has the following structure:

$$M = M_2 \to M_1 \to M_0$$

Here M_0 is an ordinary manifold, and $M_1 = E[1] = \Pi E$ for some vector bundle $E \to M_0$. Let us analyze the graded Poisson algebra of functions. The Poisson bracket is even, nondegenerate and has weight -2. Its structure is determined by the relations $\{\mathcal{A}^0, \mathcal{A}^0\} = \{\mathcal{A}^0, \mathcal{A}^1\} = 0$, $\{\mathcal{A}^1, \mathcal{A}^1\} \subset \mathcal{A}^0$, $\{\mathcal{A}^2, \mathcal{A}^0\} \subset \mathcal{A}^0$, $\{\mathcal{A}^2, \mathcal{A}^1\} \subset \mathcal{A}^1$, $\{\mathcal{A}^2, \mathcal{A}^2\} \subset \mathcal{A}^2$. The first three relations, together with the symmetry and derivation properties of the Poisson bracket, imply that E is equipped with a nondegenerate symmetric fiberwise bilinear form $\langle \cdot, \cdot \rangle$. In fact, $\mathcal{A}_1 = \Gamma(\wedge^\cdot E^*)$ is a graded Poisson subalgebra of $\mathcal{A}^\cdot$: the Poisson bracket is just an extension of $\langle \cdot, \cdot \rangle$ as a derivation in each argument. Hence, M_1 is a graded Poisson manifold whose symplectic leaves are the fibres of ΠE, and the projection $M_2 \to M_1$ is a Poisson map. One says that M_2 is a *symplectic realization* of M_1.

The rest of the relations define a Lie algebra structure on $\mathcal{A}^2$ and its action on $\mathcal{A}^0$ and $\mathcal{A}^1$. We recall that $\mathcal{A}^2$ is a locally free sheaf of $\mathcal{A}^0$-modules, i.e. the sheaf of sections of a vector bundle $\mathbb{A}$ over M_0. The Leibniz rule implies that the action of $\mathcal{A}^2$ on $\mathcal{A}^0$ comes from a bundle map $a : \mathbb{A} \to TM_0$; furthermore, the Leibniz and Jacobi identities imply that $\mathbb{A}$ is actually a Lie algebroid. By nondegeneracy, the anchor a is surjective; $\mathcal{A}^1\mathcal{A}^1 = \Gamma(\wedge^2 E^*) \subset \mathcal{A}^2$ acts trivially on $\mathcal{A}^0$, hence the kernel of a contains $\wedge^2 E^*$. Furthermore, again by Leibniz and Jacobi, the action of $\mathcal{A}^2$ on $\mathcal{A}^1$ comes from a Lie algebroid action of $\mathbb{A}$ on E, preserving $\langle \cdot, \cdot \rangle$, i.e. a Lie algebroid homomorphism from $\mathbb{A}$ into the gauge Lie algebroid of $(E, \langle \cdot, \cdot \rangle)$; by nondegeneracy, this is an isomorphism. Therefore, $\mathbb{A}$ fits into the following exact sequence of vector bundles over M_0:

$$0 \to \wedge^2 E^* \to \mathbb{A} \xrightarrow{a} TM_0 \to 0, \tag{3.1}$$

and the Lie algebroid structure on $\mathbb{A}$ is that of the gauge Lie algebroid of $(E, \langle \cdot, \cdot \rangle)$. This is the so-called *Atiyah sequence* of the pseudo-Euclidean vector bundle $(E, \langle \cdot, \cdot \rangle)$, and the Lie algebra $\mathcal{A}^2$ of sections of $\mathbb{A}$ is the *Atiyah algebra* of covariant differential operators (CDO's) on E preserving $\langle \cdot, \cdot \rangle$. The kernel $\wedge^2 E^*$ is a Lie algebra bundle, identified with $\mathfrak{so}(E)$, the endomorphisms of $(E, \langle \cdot, \cdot \rangle)$ via the Poisson bracket. We conclude that the structure of (M, Ω) is completely determined by $(E, \langle \cdot, \cdot \rangle)$.

EXAMPLE 3.2. ([**14**]) Let $A \to M_0$ be any vector bundle, and let $M = T^*[2]A[1]$. The notation means that $M = T^*\Pi A$, with the degrees assigned in such a way that the weights of a base coordinate and its conjugate momentum add up to 2 (Remark 2.1). Then the canonical symplectic form $\Omega = dp_i dq^i + d\theta_a d\xi^a$ has degree 2. It is evident that in this case $E = A \oplus A^*$, with the canonical $\langle \cdot, \cdot \rangle$ given by $\langle (X, \alpha), (Y, \beta) \rangle = \alpha(Y) + \beta(X)$.

Conversely, suppose we are given a pseudo-Euclidean vector bundle $(E, \langle \cdot, \cdot \rangle)$ over a manifold M_0. Then, of course, $M_1 = \Pi E = E[1]$ is a Poisson N-manifold (of degree -2). We shall construct its minimal symplectic realization as follows. By Example 3.2, $T^*[2]E[1]$ is the minimal symplectic realization of $(E \oplus E^*)[1]$. The map $E \hookrightarrow E \oplus E^*$ given by $X \mapsto (X, \frac{1}{2}\langle X, \cdot \rangle)$ is an isometric bundle embedding and gives rise to a map of N-manifolds $E[1] \hookrightarrow (E \oplus E^*)[1]$. We then let M be the pullback of $T[2]E[1]$, i.e. complete the commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & T^*[2]E[1] \\ \downarrow & & \downarrow \\ E[1] & \longrightarrow & (E \oplus E^*)[1] \end{array}$$

It is clear that M is a symplectic N-submanifold of $T^*[2]E[1]$, and a minimal symplectic realization of $M_1 = E[1]$. Thus we have proved

THEOREM 3.3. *N-manifolds with a symplectic structure of degree 2 are in one-to-one correspondence with pseudo-Euclidean vector bundles over ordinary manifolds. The correspondence is given by the above construction. Under this correspondence, the Lie algebra of degree-preserving canonical transformations coincides with the Atiyah algebra of the bundle.*

EXAMPLE 3.4. If the base M_0 is a point, then $M = M_1 = V[1]$ for some pseudo-Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$. In this case the degree of M is just 1. Given a basis $\{e_a\}$ of V, $\Omega = \frac{1}{2}d\xi^a g_{ab} d\xi^b$, where $g_{ab} = \langle e_a, e_b\rangle$. The algebra of symmetries in this case is $\mathcal{A}^2 = \wedge^2 V^* \simeq \mathfrak{so}(V)$, acting on $\mathcal{A}^k = \wedge^k V^*$, $k \geq 0$, via the Poisson bracket.

In the general case, a choice of local coordinates $\{x^i\}$ on M_0 and a local basis $\{e_a\}$ of sections of E such that $g_{ab} = \langle e_a, e_b\rangle$ are constant gives rise to an affine Darboux chart (q^i, ξ^a, p_i) for M: the embedding of M into $T^*[2]E[1]$ is given locally by equations $\theta_a = \frac{1}{2} g_{ab}\xi^b$, hence

$$\Omega = (dp_i dq^i + d\xi^a d\theta_a)|_M = dq^i dp_i + \frac{1}{2} d\xi^a g_{ab} d\xi^b.$$

That is, on a local chart $U \subset M_0$ such that $E|_U \simeq U \times V$ as a pseudo-Euclidean vector bundle, $M|_U \simeq T^*[2]U \times V[1]$, with Ω as above. A quadratic Hamiltonian $H = v^i(q)p_i + \frac{1}{2}\xi^a b_{ab}(q)\xi^b \in \mathcal{A}^2$ gives rise to infinitesimal transformations

$$\begin{array}{rcccl} \delta q^i & = & \{H, q^i\} & = & v^i \\ \delta \xi^c & = & \{H, \xi^c\} & = & \xi^a b_{ab}(q) g^{bc} \\ \delta p_i & = & \{H, p_i\} & = & -\frac{\partial v^j}{\partial q^i} p_j - \frac{1}{2}\xi^a \frac{\partial b_{ab}}{\partial x^i}\xi^b \end{array} \tag{3.2}$$

Integrating these, we get canonical coordinate transformations

$$\begin{array}{rcl} q^i & = & q^i(q') \\ \xi^a & = & T^a_{a'}(q')\xi^{a'} \\ p_i & = & \frac{\partial q^{i'}}{\partial q^i} p_{i'} + \frac{1}{2}\xi^{a'} \frac{\partial T^a_{a'}}{\partial x^i} g_{ab} T^b_{b'} \xi^{b'} \end{array} \tag{3.3}$$

corresponding to bundle transformations

$$\begin{array}{rcl} x^{i'} & = & x^{i'}(x) \\ e_{a'} & = & e_a T^a_{a'}(x) \end{array}$$

of E such that $T^a_{a'} g_{ab} T^b_{b'} = g_{a'b'} = \text{const}$. Notice the nonlinear (affine) term in the transformation law for p_i.

Let us now describe the global structure of the sheaf of graded $\mathcal{A}^0$-algebras $\mathcal{A}^\cdot$ in terms of $(E, \langle\cdot,\cdot\rangle)$. According to the above analysis, it is generated by $\mathcal{A}^1$ in degree 1 and $\mathcal{A}^2$ in degree 2. As sheaves on M_0, $\mathcal{A}^1 \simeq E$,[2] while $\mathcal{A}^2 \simeq \mathbb{A}$. This latter is the sheaf of infinitesimal bundle isometries of E, i.e. linear vector fields on E preserving $\langle\cdot,\cdot\rangle$, containing $\wedge^2 E \simeq \mathfrak{so}(E)$ as those vector fields that leave every point of the base fixed. Therefore,

$$\mathcal{A}^\cdot \simeq (\wedge^\cdot E \otimes S^\cdot \mathbb{A})/I,$$

[2] We identify E with E^* via $\langle\cdot,\cdot\rangle$

where I is the homogeneous ideal generated by the embedding of $\wedge^2 E$ into $\mathbb{A}$, i.e. by elements of the form $\omega \otimes 1 - 1 \otimes \omega$, $\omega \in \wedge^2 E$. Moreover, $\wedge^\cdot E$ is a sheaf of graded Poisson algebras, and so is $S^\cdot \mathbb{A}$ (because $\mathbb{A}$ is a Lie algebroid), and I is a Poisson ideal by definition of the embedding. This gives the (nondegenerate) graded Poisson algebra structure on $\mathcal{A}^\cdot$.

Each homogeneous component $\mathcal{A}^n$ has the structure of a filtered vector bundle over M_0. For $n = 2k$, $k \geq 0$, we have

$$L_k \subset L_{k-1} \subset \cdots \subset L_1 \subset L_0 = \mathcal{A}^{2k},$$

where $L_i/L_{i+1} \simeq \wedge^{2i} E \otimes S^{k-i} TM$, whereas for $n = 2k+1$, $k \geq 0$,

$$L_k \subset L_{k-1} \subset \cdots \subset L_1 \subset L_0 = \mathcal{A}^{2k+1},$$

where $L_i/L_{i+1} \simeq \wedge^{2i+1} E \otimes S^{k-i} TM_0$ (in both cases we set $L_{k+1} = 0$). Thus, $\mathcal{A}^1 = E$, $\mathcal{A}^2 = \mathbb{A}$ fits in the Atiyah sequence (3.1), $\mathcal{A}^3$ fits in the exact sequence

$$0 \to \wedge^3 E \to \mathcal{A}^3 \to E \otimes TM_0 \to 0,$$

and so on. Such a description in terms of quotients and filtered bundles is clearly unwieldy to use in practice, so one usually has to resort to local coordinates. Another alternative is to split the Atiyah sequence (3.1) by fixing a linear connection ∇ on E, preserving $\langle \cdot, \cdot \rangle$. Then $\mathcal{A}^\cdot$ can be identified with $\wedge^\cdot E \otimes S^\cdot TM_0$, with the nonzero Poisson brackets of generators $f \in C^\infty(M_0)$ (degree 0), $X, Y \in \Gamma(E)$ (degree 1) and $v, w \in \Gamma(TM)$ (degree 2) given as follows:

$$\begin{array}{rclcrcl} \{v, f\} & = & v \cdot f & & \{v, X\} & = & \nabla_v X \\ \{X, Y\} & = & \langle X, Y \rangle & & \{v, w\} & = & [v, w] + \omega(v, w) \end{array}$$

where $[\cdot, \cdot]$ denotes the Lie bracket of vector fields and $\omega \in \Omega^2(M_0, \wedge^2 E)$ is the curvature of ∇. The Jacobi identity for $\{\cdot, \cdot\}$ is then a consequence of the fact that ∇ preserves $\langle \cdot, \cdot \rangle$, and of the Bianchi identity.

4. Poisson manifolds and Courant algebroids.

Let us now turn our attention to NQ-structures on symplectic N-manifolds, i.e. homological vector fields of degree +1 preserving the symplectic form. By Lemma 2.2, such a vector field is necessarily Hamiltonian. Thus, if our manifold M is of degree n, we can speak of *Hamiltonian n-algebroids.*

Consider the case $n = 1$ first. By Proposition 3.1 $M = M_1$ is of the form $T^*[1]M_0$ for some ordinary manifold M_0, with $\Omega = d\theta_i dx^i$, and the corresponding odd Poisson bracket is just the Schouten bracket $[\cdot, \cdot]$ of multivector fields. By the above, any NQ-structure on M is determined by a quadratic Hamiltonian $\pi = -\frac{1}{2}\pi^{ij}(x)\theta_i\theta_j$ satisfying $[\pi, \pi] = 0$, i.e. a Poisson bivector field on M_0. Thus,

PROPOSITION 4.1. *Symplectic NQ-manifolds of degree* 1 *are in 1-1 correspondence with ordinary Poisson manifolds.*

The Poisson bracket of functions on M_0 can be expressed in terms of π and the Schouten bracket as the so-called *derived bracket:*

$$\{f, g\} = [[f, \pi], g]$$

By nondegeneracy of $[\cdot, \cdot]$ this completely determines π: $\pi^{ij} = \{x^i, x^j\}$. The homological Hamiltonian vector field $d_\pi = [\pi, \cdot]$ is the differential in the standard complex computing the Poisson cohomology of M_0.

Consider now the case $n = 2$. By Theorem 3.3, such a manifold (M, Ω) corresponds to a pseudo-Euclidean vector bundle E over some ordinary manifold M_0. By Lemma 2.2, an NQ-structure on M is determined by a *cubic* Hamiltonian Θ ("BRST charge") satisfying the *structure equation*

$$\{\Theta, \Theta\} = 0 \tag{4.1}$$

where $\{\cdot, \cdot\}$ is the even Poisson bracket on M of degree -2. One should expect such a Θ to correspond to some structure on E, recoverable via derived brackets. This structure is called a *Courant algebroid.* It was first defined in [**9**]. We give an equivalent definition from [**14**]:

DEFINITION 4.2. A *Courant algebroid* is a pseudo-Euclidean vector bundle $(E, \langle \cdot, \cdot \rangle)$ over a manifold M_0, together with a bilinear operation $\circ$ on $\Gamma(E)$ and a bundle map $a : E \to TM_0$ (the anchor), satisfying the following properties:

(1) $e \circ (e_1 \circ e_2) = (e \circ e_1) \circ e_2 + e_1 \circ (e \circ e_2) \quad \forall e, e_1, e_2 \in \Gamma(E)$;
(2) $a(e_1 \circ e_2) = [a(e_1), a(e_2)] \quad \forall e_1, e_2 \in \Gamma(E)$;
(3) $e_1 \circ (fe_2) = f(e_1 \circ e_2) + (a(e_1) \cdot f)e_2 \quad \forall e_1, e_2 \in \Gamma(E),\ f \in C^\infty(M_0)$;
(4) $\langle e, e_1 \circ e_2 + e_2 \circ e_1 \rangle = a(e) \cdot \langle e_1, e_2 \rangle \quad \forall e, e_1, e_2 \in \Gamma(E)$;
(5) $a(e) \cdot \langle e_1, e_2 \rangle = \langle e \circ e_1, e_2 \rangle + \langle e_1, e \circ e_2 \rangle \quad \forall e, e_1, e_2 \in \Gamma(E)$.

One can think of sections of E as acting on E by "left multilications" $e\circ$. Properties (1), (3) and (5) mean that this action preserves the whole structure, while property (2) says that the anchor is equivariant with respect to this action and the Lie derivative action on TM_0. Property (1) defines a Leibniz algebra structure on $\Gamma(E)$; if $\circ$ were skew-symmetric, it would be a Lie algebra. Property (4) says that the symmetric part of $\circ$ is "infinitesimal". If we define the *co-anchor* $a^* : T^*M_0 \to E$ by $\langle a^*\nu, e \rangle = \nu(a(e))$, and set $D = a^*d$, property (4) reads:

$$e_1 \circ e_2 + e_2 \circ e_1 = D\langle e_1, e_2 \rangle$$

Together with property (2), this implies the useful identity

$$aa^* = 0 \tag{4.2}$$

implying that the image of a^* is isotropic. Other useful identities implied by the definition are

$$\begin{aligned} e \circ Df &= D\langle e, Df \rangle \\ Df \circ e &= 0 \end{aligned} \tag{4.3}$$

and more generally

$$\begin{aligned} e \circ a^*\nu &= a^* L_{a(e)} \nu \\ a^*\nu \circ e &= -a^* \iota_{a(e)} d\nu \end{aligned} \tag{4.4}$$

EXAMPLE 4.3. Let $M_0 =$ point. Then E is just a pseudo-Euclidean vector space, and a Courant algebroid structure is just a Lie algebra such that $\langle \cdot, \cdot \rangle$ is ad-invariant. More generally, any Courant algebroid with $a = 0$ is a bundle of such Lie algebras. By (4.2), this must always be the case if $\langle \cdot, \cdot \rangle$ is definite.

EXAMPLE 4.4. Let $E = TM_0 \oplus T^*M_0$ with the canonical $\langle \cdot, \cdot \rangle$ as in Example 3.2. Define a to be the projection onto TM_0, and $\circ$ by

$$(v, \xi) \circ (w, \eta) = ([v, w], L_v\eta - \iota_w d\xi) \tag{4.5}$$

This operation (rather, its skew-symmetrization) was used originally by T. Courant [**2**] to describe integrable Dirac structures on M_0, including closed 2-forms, Poisson

bivectors and foliations. This example is responsible for the term "Courant algebroid", coined by the authors of [9] who considered generalizations of this example to dual pairs of Lie algebroids.

THEOREM 4.5. *Symplectic NQ-manifolds of degree* 2 *are in 1-1 correspondence with Courant algebroids.*

PROOF. Let (M, Ω) be the symplectic N-manifold corresponding to $(E, \langle\cdot,\cdot\rangle)$, with $(\mathcal{A}^\cdot, \{\cdot,\cdot\})$ its graded Poisson algebra of polynomial functions. Then $\mathcal{A}^0$ is identified with $C^\infty(M_0)$, $\mathcal{A}^1$ – with $\Gamma(E)$, and $\{\cdot,\cdot\}$ restricted to $\mathcal{A}^1$ is just $\langle\cdot,\cdot\rangle$. Let $\Theta \in \mathcal{A}^3$ satisfy the structure equation (4.1). Given arbitrary $f \in \mathcal{A}^0$ and $e, e_1, e_2 \in \mathcal{A}^1$, define a and $\circ$ as the derived brackets

$$\begin{aligned} a(e)\cdot f &= \{\{e,\Theta\},f\} = \{\{\Theta,f\},e\} \\ e_1 \circ e_2 &= \{\{e_1,\Theta\},e_2\} \end{aligned} \tag{4.6}$$

The proof that this defines a Courant algebroid is exactly the same as the proof of Theorem 3.7.3 in [14], which asserted the same statement for $E = A \oplus A^*$, $M = T^*[2]A[1]$. But the splitting plays no role in the argument, so we will not repeat it here. It is a straightforward derivation using the basic properties of $\{\cdot,\cdot\}$ and the structure equation.

Due to the nondegeneracy of $\{\cdot,\cdot\}$, the above derived brackets uniquely determine Θ. In an affine Darboux chart (q^i, ξ^a, p_i) on M, corresponding to a chart $\{x^i\}$ on M_0 and a local basis $\{e_a\}$ of sections of E such that $\langle e_a, e_b\rangle = g_{ab} = \text{const.}$, $\Omega = dp_i dq^i + \frac{1}{2} d\xi^a g_{ab} d\xi^b$, and $\Theta \in \mathcal{A}^3$ is of the form

$$\Theta = \xi^a A_a^i(q) p_i - \frac{1}{6}\phi_{abc}(q)\xi^a\xi^b\xi^c \tag{4.7}$$

It clearly follows from (4.6) that

$$\begin{aligned} A_a^i &= a(e_a)\cdot x^i \\ \phi_{abc} &= \langle e_a \circ e_b, e_c\rangle \end{aligned} \tag{4.8}$$

where $e_a = g_{ab}\xi^b$. So, conversely, given a Courant algebroid structure on $(E, \langle\cdot,\cdot\rangle)$, we can define Θ in a Darboux chart by the formulas (4.7), (4.8). It is an easy check, using the coordinate transformations (3.3), that Θ is in fact globally defined and obeys (4.1) due to the properties of a Courant algebroid. In fact,

$$\begin{aligned} \{\Theta,\Theta\} &= \langle a^* dx^i, a^* dx^j\rangle p_i p_j + \xi^a\xi^b dx^j([a(e_a), a(e_b)] - a(e_a\circ e_b))p_j + \\ &+ \tfrac{1}{12}\langle (e_a\circ e_b)\circ e_c + e_b\circ(e_a\circ e_c) - e_a\circ(e_b\circ e_c), e_d\rangle \xi^a\xi^b\xi^c\xi^d \end{aligned}$$

So $\{\Theta,\Theta\} = 0$ if and only if $(E, \langle\cdot,\cdot\rangle, a, \circ)$ is a Courant algebroid. □

EXAMPLE 4.6. If M_0 is a point, $\Theta \in \wedge^3 E^*$ is the Cartan structure tensor corresponding to a Lie algebra structure on E making $\langle\cdot,\cdot\rangle$ ad-invariant: $\Theta(X,Y,Z) = \langle [X,Y], Z\rangle$.

EXAMPLE 4.7. For $E = TM_0 \oplus T^*M_0$ with the standard Courant algebroid structure, $M = T^*[2]T[1]M_0$, and $\Theta = \Theta_0$ is the Hamiltonian corresponding to the de Rham vector field d on $T[1]M_0$. This example will be studied in detail in the next section.

EXAMPLE 4.8. More generally, for $E = A \oplus A^*$, the double of a Lie bialgebroid (A, A^*) [9], $M = T^*[2]A[1] \simeq T^*[2]A^*[1]$, and $\Theta = \mu + \gamma$ where μ and γ are the commuting Hamiltonians defining the Lie bialgebroid structure [14].

EXAMPLE 4.9. In particular, let a Lie algebra $\mathfrak{g}$ act on a manifold M_0 (on the right). Endow the trivial bundle $M \times \mathfrak{g}$ with the transformation (semidirect product) Lie algebroid structure, the dual bundle with the zero structure, and let $E = M \times (\mathfrak{g} \oplus \mathfrak{g}^*)$ be the double. Then $M = (\mathfrak{g} \oplus \mathfrak{g}^*)[1] \times T^*[2]M_0$, and the BRST charge is the classical one:

$$\Theta = \xi^a v_a^i(q) p_i - \frac{1}{2}\xi^a \xi^b C_{ab}^c \theta_c$$

where $v_a = v_a^i(x)\frac{\partial}{\partial x^i}$ are the infinitesimal generators of the action corresponding to a basis (e_a) of $\mathfrak{g}$, and (C_{ab}^c) are the structure constants in this basis. Notice that our grading is different from the "ghost degree" grading. This leads to a different grading in the cohomology as well, only the parity remains the same. However, under sufficient regularity assumption, the total cohomology is isomorphic to the algebra of functions on the reduced phase space, as proved in [**7**], so this change in grading is of no consequence.

Given a Courant algebroid, $(\mathcal{A}^{\cdot}, D = \{\Theta, \cdot\})$ becomes a differential complex, the *standard complex* of $(E, \langle\cdot,\cdot\rangle, a, \circ)$. Since D is a Hamiltonian vector field and M has degree 2, it is completely determined by its action on $\mathcal{A}^0 = C^\infty(M_0)$ and $\mathcal{A}^1 = \Gamma(E)$. In fact, the formulas (4.6) imply that $D : C^\infty(M_0) \to \Gamma(E)$ is just $D = a^*d$, while $D : \Gamma(E) \to \Gamma(\mathbb{A})$ is given by

$$(De_1) \cdot e_2 = e_1 \circ e_2$$

where $\cdot$ denotes the Lie algebroid action of $\Gamma(\mathbb{A})$ on $\Gamma(E)$. One interesting feature of this complex is that it is infinite in general.

The cohomology groups $H^{\cdot}(M, D)$ in low degrees have familiar structural interpretations. Thus, H^0 is the space of smooth functions on M_0 that are constant along the leaves of the anchor foliation; it is equal to $\mathbb{R}$ for transitive Courant algebroids (those with surjective anchor). H^1 is the space of sections of E acting trivially on E, modulo those of the form Df for some function f. Further, H^2 is the space of linear vector fields on E preserving the Courant algebroid structure, modulo those generated by sections of E as $e\circ$. H^3 is the space of infinitesimal deformations of the Courant algebroid structure, modulo the trivial ones generated by $\Gamma(\mathbb{A})$, while H^4 houses the obstructions to extending an infinitesimal deformation to a formal one. For the purposes of deformation theory it is customary to shift the grading by -2, the degree of $\{\cdot,\cdot\}$, so that H^3 and H^4 become, respectively, H^1 and H^2. Notice the analogy with Poisson manifolds, and with Lie algebras.

5. Higher de Rham complexes and exact Courant algebroids .

Let us consider the case $E = TM_0 \oplus T^*M_0$. In this case $M = T^*[2]T[1]M_0 = T^*\Pi TM_0$, as in Example 3.2. Any local coordinates (x^i) on M_0 give rise to affine Darboux coordinates $(q^i, \xi^i, p_i, \theta_i)$, so that $\Omega = dq^i dp_i + d\xi^i d\theta_i$. This manifold has a rich additional structure. First of all, there is a canonical isomorphism, the *Legendre transformation*, that identifies $T^*\Pi TM_0$ with $T^*\Pi T^*M_0$ by exchanging the momenta conjugate to the fibre coordinates on ΠTM_0 with the fibre coordinates on ΠT^*M_0, and vice versa. This holds for any vector bundle $A \to M_0$ [**14**]. Thus, M is what is known as a *double vector bundle*:

$$\begin{array}{ccc} M & \longrightarrow & T^*[1]M_0 \\ \downarrow & & \downarrow \\ T[1]M_0 & \longrightarrow & M_0 \end{array}$$

This means that each arrow is a vector bundle; the horizontal pair of arrows is a morphism of the vertical vector bundles, while the vertical pair is a morphism of the horizontal bundles. Because of this structure, the algebra $\mathcal{A}^{\cdot}$ of polynomial functions on M acquires a double grading; we shall use the notation $\mathcal{A}^{\cdot,\cdot} = \oplus_{k,l\geq 0}\mathcal{A}^{k,l}$. The Euler vector fields corresponding to the two gradings are, respectively,

$$\epsilon_1 = p_i\frac{\partial}{\partial p_i} + \theta_i\frac{\partial}{\partial \theta_i}$$

and

$$\epsilon_2 = p_i\frac{\partial}{\partial p_i} + \xi^i\frac{\partial}{\partial \xi^i}$$

The N-manifold structure on $M = T^*[2]T[1]M_0 = T^*[2]T^*[1]M_0$ corresponds to the total grading

$$\epsilon = \epsilon_1 + \epsilon_2 = 2p_i\frac{\partial}{\partial p_i} + \xi^i\frac{\partial}{\partial \xi^i} + \theta_i\frac{\partial}{\partial \theta_i}$$

Evidently, the canonical sympelctic form Ω has bi-degree $(1,1)$, the corresponding Poisson bracket – $(-1,-1)$.

There is another special canonical isomorphism: $T^*[2]T^*[1]M_0 = T^*\Pi T^*M_0 \simeq \Pi T\Pi T^*M_0 = T[1]T^*[1]M_0$. It comes from treating the conjugate momenta p_i, ξ^i as the differentials of the basic coordinates: $p_i = d\theta_i$, $\xi^i = dx^i$.[3] Correspondingly, on M there is the canonical (de Rham) vector field

$$D = \xi^i\frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial \theta_i}$$

This vector field is homological and Hamiltonian: $D = \{\Theta_0, \cdot\}$, where $\Theta_0 = \xi^i p_i \in \mathcal{A}^{1,2}$ obeys (4.1). With respect to the fibration $M = T^*\Pi TM_0 \to \Pi TM_0$, it is just the Hamiltonian lift of the de Rham vector field $d = \xi^i\frac{\partial}{\partial x^i}$ on $T[1]M_0$. With respect to the other fibration $M = T^*\Pi T^*M_0 \to \Pi T^*M_0$, Θ_0 is quadratic and generates the canonical Schouten bracket on $T^*[1]M_0$ as a derived bracket:

$$[X,Y] = \{\{X,\Theta_0\},Y\} \tag{5.1}$$

for multivector fields X, Y.

The vector field D has bi-degree $(0,+1)$. This can be expressed by the relations

$$\begin{aligned}[\epsilon_1, D] &= 0\\ [\epsilon_2, D] &= D\end{aligned}$$

This means that D preserves the subspace $\mathcal{A}^{k,\cdot}$ for each $k \geq 0$, and the restriction to each $\mathcal{A}^{k,\cdot}$ is a differential. For $k = 0$, $(\mathcal{A}^{0,\cdot}, D) = (\Omega^{\cdot}(M_0), d)$, the de Rham complex of M_0; we call the complexes $(\mathcal{A}^{k,\cdot}, D)$ for $k > 0$ the *higher de Rham complexes*. In particular, $\mathcal{A}^{1,\cdot}[1]$ is naturally identified with vector fields on $T[1]M_0$, i.e. derivations of $\Omega^{\cdot}(M_0)$, with $D = [d, \cdot]$. In general, $\mathcal{A}^{k,l}$ is the space of symbols of differential operators on $T[1]M_0$ of order k and degree $l - k$.

What is important is that ϵ_1, viewed as a cocycle with respect to $[D, \cdot]$, is actually a coboundary. That is, there exists a vector field ι on M, of bi-degree $(0,-1)$, such that

$$[D, \iota] = D\iota + \iota D = \epsilon_1 \tag{5.2}$$

[3] For a thorough treatment of double vector bundle and canonical isomorphisms (without parity shifts), see [5]

In coordinates,

$$\iota = \theta_i \frac{\partial}{\partial p_i}$$

It is clear that ι satisfies the above relation, and is homological. With respect to the symplectic structure Ω on M, ι is not Hamiltonian, but rather satisfies the relation:

$$L_\iota \Omega + \omega = 0,$$

where $\omega = d\theta_i dx^i$ is the canonical odd symplectic form on $T^*[1]M_0$, pulled back to M. This relates symplectic N-manifolds of degrees 1 and 2.

To show that ι is well-defined, one can check that it is invariant under coordinate transformations, but we shall provide a coordinate-free description instead. It suffices to describe the action of ι on $\mathcal{A}^{0,\cdot}$ and $\mathcal{A}^{1,\cdot}$. On $\mathcal{A}^{0,\cdot}$ it acts trivially; let $V \in \mathcal{A}^{1,l}$. Then V is a derivation of $\Omega^\cdot(M_0)$ of degree $l-1$. The restriction of V to $\Omega^0(M_0) = C^\infty(M_0)$ is a derivation with values in $\Omega^{l-1}(M_0)$, i.e. a $(l-1)$-form-valued vector field $\bar{V}$ on M_0. We set $\iota(V) = \iota_{\bar{V}}$, the contraction operator. Now, ϵ_1 acts on $\mathcal{A}^{1,\cdot}$ as the identity operator. Therefore, by (5.2), the operators ιD and $D\iota$ are complementary projections. This leads to the following decomposition:

$$\mathcal{A}^{1,\cdot}[1] = \mathcal{A}^{1,\cdot+1} = \mathrm{Der}^\cdot(\Omega(M_0)) \simeq \mathcal{G}[\tau] = \mathcal{G}^\cdot[-1] \oplus \mathcal{G}^\cdot \tag{5.3}$$

as graded vector spaces. Here $\mathcal{G}^\cdot = \Gamma(\wedge^\cdot T^* M_0 \otimes TM_0)$, the space of vector-valued differential forms, and $\mathcal{G}[\tau]$ denotes the adjunction of a formal odd variable of degree -1; the differential $D = [d, \cdot]$ acts as $\frac{d}{d\tau}$. The space $\mathcal{G}^\cdot$ appears in the first summand as "contractions", in the second – as "Lie derivatives". The above decomposition (5.3) is a classic result of Frölicher and Nijenhuis [**3**]. By viewing D and ι as vector fields on M, we obtain the following generalization:

PROPOSITION 5.1. *The higher de Rham complexes $(\mathcal{A}^{k,\cdot}, D)$, $k > 0$, are acyclic.*

PROOF. Immediate from (5.2), since ϵ_1 acts on $\mathcal{A}^{k,\cdot}$ as multiplication by k. This also gives rise to a decomposition of $\mathcal{A}^{k,\cdot}$ generalizing (5.3). □

The above proposition has several consequences. For example, in terms of the standard Lie algebroid structure on TM_0, the de Rham complex $(\mathcal{A}^{0,\cdot}, d)$ is the standard complex with coefficient in the trivial module, whereas the higher complex $(\mathcal{A}^{k,\cdot}, D)$ should be viewed as "the standard complex with coefficients in the $k$the exterior power of the adjoint module", by analogy with Lie algebras. Thus, for instance, the complex $(\mathcal{A}^{1,\cdot}[1], [d, \cdot])$ is responsible for the deformation theory of this algebroid, and its acyclicity implies the rigidity of the standard Lie algebroid:

COROLLARY 5.2. *The standard Lie algebroid structure on TM_0 is rigid.*

The complex $(\mathcal{A}^{2,\cdot}, D)$ is also interesting. As shown in [**14**], 1-cocycles in that complex are responsible for Lie algebroid structures on T^*M_0, compatible with the standard one on TM_0 in the sense that (TM_0, T^*M_0) is a Lie bialgebroid [**11**]. More precisely, a $\gamma \in \mathcal{A}^{2,1}$ gives rise to a Lie algebroid on T^*M_0 if and only if $\{\gamma, \gamma\} = 0$, and it is compatible with TM_0 if and only if $D\gamma = \{\Theta_0, \gamma\} = 0$. By Proposition 5.1, $\gamma = D\pi = \{\Theta_0, \pi\}$, where $\pi \in \mathcal{A}^{2,0} = \Gamma(\wedge^2 TM_0)$ is some bivector field. Then by (5.1), $\{\gamma, \gamma\} = 0$ is equivalent to $\{\Theta_0, [\pi, \pi]\} = 0$; by Proposition 5.1 (with $k = 3$), this can only happen if $[\pi, \pi] = 0$. Therefore, we have another

COROLLARY 5.3. *Any Lie algebroid structure on T^*M_0 compatible with the standard one on TM_0 comes from a Poisson structure on M_0.*

Finally, let us consider (M, Ω, Θ_0) as a symplectic NQ-manifold. By Theorem 4.5, it must corresond to a Courant algebroid structure on $E = TM_0 \oplus T^*M_0$; it is easy to see that this Courant algebroid is the standard one, as in Example 4.4. Let us denote this structure by E_0. The standard complex of E_0 is the total complex $(\mathcal{A}^\cdot, D)$. The Proposition 5.1 immediately implies

COROLLARY 5.4. *The cohomology of the standard complex of the Courant algebroid E_0 is isomorphic to the de Rham cohomology of M_0.*

We can describe the action of the de Rham cohomology explicitly. Thus, 1-forms on M_0 act on any Courant algebroid over M_0 via (4.4), and closed 1-forms act trivially; in case of E_0, they are the only sections that do. Furthermore, a 2-form $\beta \in \mathcal{A}^{0,2} \subset \mathcal{A}^2$ acts by changing the isotropic splitting of $E = TM_0 \oplus T^*M_0$ via its graph: $(v, \xi) \mapsto (v, \xi + \iota_v \beta)$. This does not change $\langle \cdot, \cdot \rangle$ and a, but $\circ$ in (4.5) is modified by the term $d\beta(v, w, \cdot)$. So the Courant algebroid structure is preserved if $d\beta = 0$; if $\beta = d\alpha$, the action is $\alpha\circ$, as in (4.4).

Lastly, $H^3(M_0, \mathbb{R})$ corresponds to nontrivial deformations of E_0. A 3-form ϕ acts by adding the term $\phi(v, w, \cdot)$ to the expression (4.5) for $\circ$. The corresponding structure is

$$\Theta_\phi = \Theta_0 - \phi = \xi^i p_i - \frac{1}{6}\phi_{ijk}(q)\xi^i\xi^j\xi^k$$

where $\phi = \frac{1}{6}\phi_{ijk}(x)dx^i \wedge dx^j \wedge dx^k$. It is easy to check that Θ_ϕ obeys the structure equation (4.1) if and only if $d\phi = 0$. Let us denote the corresponding Courant algebroid E_ϕ. Cohomologous 3-forms give rise to isomorphic Courant algebroids: if $\phi' - \phi = d\beta$, changing the splitting by β maps E_ϕ to $E_{\phi'}$. In particular, exact 3-forms yield trivial deformations of E_0. Corollary 5.4 implies that these E_ϕ's are the only possible deformations of E_0.

In effect, we have recovered, via deformation theory, Ševera's classification of so-called *exact Courant algebroids* [**18**]. These are Courant algebroids such that the sequence

$$0 \to T^*M_0 \xrightarrow{a^*} E \xrightarrow{a} TM_0 \to 0$$

is exact. Choosing an isotropic splitting $\sigma : TM_0 \to E$ (a "connection") identifies E with E_ϕ, where ϕ is the "curvature" of σ, given by $\phi(u, v, w) = \langle \sigma(u) \circ \sigma(v), \sigma(w) \rangle$. The cohomology class of ϕ is a characteristic class, the *Ševera class* of E.

EXAMPLE 5.5. (due to Anton Alexeev) As an example of an exact Courant algebroid with a nontrivial Ševera class, consider a Lie group G with a bi-invariant metric K. Let $E = (\mathfrak{g} \oplus \mathfrak{g}) \times G$, where $\mathfrak{g} = \mathrm{Lie}(G)$. Define $\langle \cdot, \cdot \rangle$ by

$$\langle (X, Y), (X', Y') \rangle = K(X, X') - K(Y, Y')$$

and an anchor $a : E \to TG$ by

$$a(X, Y) = X^l - Y^r$$

i.e. $a(X, Y, g) = gX - Yg$. Finally, define $\circ$ on constant sections by

$$(X, Y) \circ (X', Y') = ([X, X'], [Y, Y'])$$

By trivializing both TG and T^*G as $G \times \mathfrak{g}$ by left translation and using K, it is easy to see that this defines an exact Courant algebroid over G. Let us split a by $\sigma(Z, g) = \frac{1}{2}(Z, -\mathrm{Ad}_g Z)$. One computes the curvature to be $\phi(X, Y, Z) = -\frac{1}{2}K([X, Y], Z)$, the Cartan structure tensor. Thus, the Ševera class of this Courant algebroid is the canonical class of G.

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Department of Mathematics, Pennsylvania State University, University Park, PA 16802

E-mail address: roytend@math.psu.edu

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On certain canonical diffeomorphisms in symplectic and Poisson geometry

K. C. H. Mackenzie

ABSTRACT. Associated with the canonical symplectic structure on a cotangent bundle T^*M is the diffeomorphism $\#\colon T^*(T^*M) \longrightarrow T(T^*M)$. This and the Tulczyjew diffeomorphism $T(T^*M) \longrightarrow T^*(TM)$ may be derived from the canonical involution $T(TM) \longrightarrow T(TM)$ by suitable dualizations. We show that the constructions which yield these maps extend very generally to the double Lie algebroids of double Lie groupoids, where they play a crucial role in the relations between double Lie algebroids and Lie bialgebroids.

There have been several talks this meeting about notions of double for Lie bialgebroids. Some of these have derived from the 1997 construction of Liu Zhang-Ju, Alan Weinstein and Xu Ping [**4**] in which they introduced the notion of Courant algebroid, and some have involved elements of super mathematics. I very much hope that before long there will be a clear account of the relations between these various approaches and even a unification of them.

There is another approach to the question of doubles, which was not at first related to Lie bialgebroids, but arose out of broad considerations of what may be called "second–order geometry". It is not possible to describe this approach from scratch in an hour, but it is appropriate at this conference to indicate the broad features of it. This talk therefore takes a slice through the papers [**5**], [**7**], [**6**], [**8**], transverse to their chronological sequence and provides an alternative route to approach them. Some proofs in §2 and §4 are new.

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1

The three canonical diffeomorphisms with which the paper begins are associated with iterated tangent and cotangent bundles:

(1) The canonical involution on an iterated tangent bundle

$$J\colon T^2M \longrightarrow T^2M,$$

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which can be loosely described as 'interchanging the order of differentiation'.

Regard elements of $T^2M = T(TM)$ as represented by equivalence classes of squares $\varphi\colon I^2 \longrightarrow M$. Thus $\varphi = \varphi(s,t)$ differentiates to a curve of tangent vectors $X_t = \frac{\partial \varphi}{\partial s}(0,t)$ and X_t in turn differentiates to $\xi = \frac{d}{dt}X_t|_0 \in T_{X_0}(TM)$. Now J is induced by the map $\varphi \longmapsto \varphi \circ \sigma$ where $\sigma(s,t) = (t,s)$.

(2) The Poisson anchor for the canonical symplectic structure on a cotangent bundle T^*M:

$$\pi^{\#}_{d\theta}\colon T^*(T^*M) \longrightarrow T(T^*M).$$

Any Poisson structure on a manifold P induces a map $\#\colon T^*P \longrightarrow TP$ which sends an exact differential δf to the Hamiltonian vector field X_f. This map serves as the anchor of the Lie algebroid structure defined by the bracket of 1–forms. When P is the cotangent bundle $c\colon T^*M \longrightarrow M$, the canonical symplectic structure is $\omega = \delta\theta$ where θ is the 1–form which associates to $\eta \in T_\alpha(T^*M)$ the pairing of α with $T(c)(\eta) \in TM$.

(3) The canonical diffeomorphism

$$\Theta\colon T(T^*M) \longrightarrow T^*(TM)$$

defined intrinsically by Tulczyjew in [**12**, p.97] and locally in, for example, [**1**, p.424]. We call this the *Tulczyjew diffeomorphism.* Giving $T(T^*M)$ the tangent lift of the symplectic structure on T^*M, this Θ is a symplectic diffeomorphism to the canonical structure on $T^*(TM)$.

Anyone who meets these diffeomorphisms for the first time must surely suspect that they are related. For example, one would like (2) and (3) to be duals of (1). However, it is not at first clear what one might mean by this.

The best way to understand these maps is to extend them into a more general framework.

2

In this section we consider those aspects of the constructions which do not depend upon the bracket of vector fields or the dual symplectic structure.

First consider T^2M. It has two vector bundle structures, the standard structure as the tangent bundle of TM, for which the projection is denoted p_{TM}, and what we call the *prolongation* structure, with projection $T(p_M)$ and operations which are obtained by applying the tangent functor T to the operations in TM. These two structures are represented in the diagram:

$$\begin{array}{ccc} & \overset{T(p_M)}{} & \\ T^2M & \longrightarrow & TM \\ {\scriptstyle p_{TM}}\downarrow & & \downarrow \\ TM & \longrightarrow & M \end{array}$$

It is easier to keep the role of the two structures clear in one's mind if one replaces TM by an arbitrary vector bundle (E, q, M):

(1)
$$\begin{array}{ccc} & \overset{T(q)}{} & \\ TE & \longrightarrow & TM \\ {\scriptstyle p_E}\downarrow & & \downarrow \\ E & \longrightarrow & M \end{array}$$

Question: If we dualize $TE \longrightarrow E$, is there a second structure which completes the square in

$$\begin{array}{ccc} T^*E & & ? \\ \downarrow & & \\ E & \longrightarrow & M \end{array}$$

To see the answer, consider the two short exact sequences associated with TE:

$$q^!E \rightarrowtail TE \twoheadrightarrow q^!TM$$
$$p_M^!E \rightarrowtail TE \twoheadrightarrow p_M^!E$$

Here shrieks denote pullback bundles. The first sequence is a short exact sequence of vector bundles over base E and the surjection is the map induced by $T(q)$. The second sequence is over TM and the surjection is induced by p_E. The two surjections are actually the same map $TE \longrightarrow E \times_M TM$, but are considered with respect to different bundle structures.

The kernel of the first surjection is the vertical bundle of q; that is, the bundle of vectors tangent to the fibres of E. Such a vector is determined by its base point and by a vector in E itself to which it is parallel, and the injection $q^!E \longrightarrow TE$ maps (e_1, e_2) to the vertical tangent vector with base point e_1 and parallel to e_2.

The kernel of the second surjection consists of the tangent vectors to E along the zero section. Such a vector ξ may be written as $T(0)(x)+e$ where $x = T(q)(\xi) \in T_m(M)$ and $e \in E_m$ is identified with the vertical vector based at 0_m and parallel to e.

Thus the E which occurs in both kernels is precisely the set of tangent vectors to E which are vertical and based at points of the zero section; that is, it is $\ker(T(q)) \cap \ker(p_E)$.

Dualizing the first short exact sequence gives

$$q^!T^*M \rightarrowtail T^*E \twoheadrightarrow q^!E^*$$

and this suggests that the answer to the question is

$$\begin{array}{ccc} T^*E & \longrightarrow & E^* \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array} \tag{2}$$

That this is indeed so was proved in [**9**].

To find the dual of the other structure, $TE \longrightarrow TM$, apply T to the pairing $E \times_M E^* \longrightarrow \mathbb{R}$. This gives a pairing $TE \times_{TM} T(E^*) \longrightarrow \mathbb{R}$ which is again non–degenerate. So

$$(TE \longrightarrow TM)^* \cong (T(E^*) \longrightarrow TM) \tag{3}$$

and so dualizing the horizontal structure in (1) gives

$$\begin{array}{ccc} T(E^*) & \longrightarrow & TM \\ \downarrow & & \downarrow \\ E^* & \longrightarrow & M \end{array} \tag{4}$$

This is precisely the structure (1) with E replaced by E^*.

Now consider more general structures of this type.

A *double vector bundle* ([**10**]; see also [**5**, §1]) is a manifold D with two vector bundle structures, on bases A and B, each of which is a vector bundle on base M,

$$\begin{array}{ccc} D & \overset{\widetilde{q}_H}{\longrightarrow} & B \\ {\scriptstyle \widetilde{q}_V}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \underset{q_A}{\longrightarrow} & M \end{array}$$

such that the structure maps of each vector bundle structure on D are vector bundle morphisms with respect to the other. Precisely: for the horizontal structure,

- the projection $\widetilde{q}_H \colon D \longrightarrow B$ is a vector bundle morphism over the projection $q_A \colon A \longrightarrow M$;
- the zero section $B \longrightarrow D$ is a vector bundle morphism over the zero section $M \longrightarrow A$;
- the addition $D \times_B D \longrightarrow D$ is a vector bundle morphism over the addition $A \times_M A \longrightarrow A$.
- the scalar multiplication $D \times \mathbb{R} \longrightarrow D$ is a vector bundle morphism over the scalar multiplication $A \times \mathbb{R} \longrightarrow A$.

The first three conditions are precisely those which ensure that for four elements $d_1, d_2, d_3, d_4 \in D$ such that the LHS of

$$(d_1 \underset{H}{+} d_2) \underset{V}{+} (d_3 \underset{H}{+} d_4) = (d_1 \underset{V}{+} d_2) \underset{H}{+} (d_3 \underset{V}{+} d_4)$$

is defined, the RHS is defined and the two sides are equal. That (1) is indeed a double vector bundle is easily verified.

Now consider the intersection of the two kernels

$$C = \ker(\widetilde{q}_H) \cap \ker(\widetilde{q}_V).$$

A priori, there is no reason to expect this to have a natural algebraic structure. However, each structure on D induces a vector bundle structure on C with base M, and these two structures coincide. Pradines' terminology in [**10**] was *coeur*; we call it the *core* of D .

There is again a short exact sequence

$$q_A^! C \rightarrowtail D \twoheadrightarrow q_A^! B$$

of vector bundles over A and dualizing this gives

$$q_A^! B^* \rightarrowtail D^{*V} \twoheadrightarrow q_A^! C^*$$

and we get another double vector bundle

$$\begin{array}{ccc} D^{*V} & \longrightarrow & C^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

called the *vertical dual* of D. Likewise there is a *horizontal dual*

$$\begin{array}{ccc} D^{*H} & \longrightarrow & B \\ \downarrow & & \downarrow \\ C^* & \longrightarrow & M \end{array}$$

Theorem 1: ([**6**], [**3**]) The vector bundles $D^{*V} \longrightarrow C^*$ and $D^{*H} \longrightarrow C^*$ are themselves dual under a pairing which is canonical up to sign and which respects the double vector bundle structures.

To define the pairing, consider elements $\Phi \in D^{*V}$ and $\Psi \in D^{*H}$, shown with their various projections as follows:

$$\begin{array}{ccc} \Phi & \longmapsto & \kappa \\ \downarrow & & \downarrow \\ a & \longmapsto & m \end{array} \qquad \begin{array}{ccc} \Psi & \longmapsto & b \\ \downarrow & & \downarrow \\ \kappa & \longmapsto & m \end{array}$$

where $a \in A$, $b \in B$ and $\kappa \in C^*$. The given elements must of course lie over the same point of C^*. Now given a and b over the same $m \in M$, there exist elements

$d \in D$ of the form

$$\begin{array}{ccc} d & \longmapsto & b \\ \downarrow & & \downarrow \\ a & \longmapsto & m \end{array}$$

and we define

$$\langle \Phi, \Psi \rangle_{C^*} = \langle \Phi, d \rangle_A - \langle \Psi, d \rangle_B$$

where the subscripts indicate the bases of the pairings.

We could equally well reverse the order on the RHS. The statement in the theorem that this duality respects the double vector bundle structures means that the induced isomorphism from the dual of $D^{*V} \longrightarrow C^*$ to $D^{*H} \longrightarrow C^*$ is in fact an isomorphism of double vector bundles.

Example: Consider $D = TE$ as in (1). We observed that the two duals are $D^{*V} = T^*E$ as in (2) and $D^{*H} = T(E^*)$ as in (4). So Theorem 1 implies that there is a non–degenerate pairing of the two bundles

$$T^*E \longrightarrow E^* \qquad \text{and} \qquad T(E^*) \longrightarrow E^*.$$

To recognize what this is, write $A = E^*$. Then the statement is that the two bundles

$$T^*(A^*) \longrightarrow A \qquad \text{and} \qquad T^*(A) \longrightarrow A$$

are isomorphic as double vector bundles. That this is so was proved (by different methods) in [**9**]. To state the result precisely: Given a vector bundle $A \longrightarrow M$, there is a canonical isomorphism of double vector bundles $R_A \colon T^*(A^*) \longrightarrow T^*(A)$ over A and A^*, which is an anti–symplectomorphism and which induces the negative of the identity on the cores $T^*M \longrightarrow T^*M$.

A proof of the next result is also in [**9**].

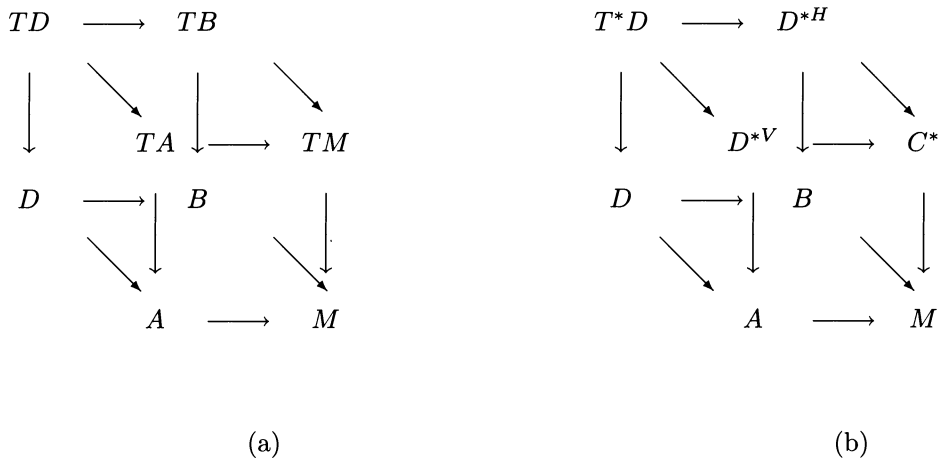
Proposition: For any manifold M, the composite

$$T^*(T^*M) \xrightarrow{\pi^{\#}_{d\theta}} T(T^*M) \xrightarrow{\Theta} T^*(TM)$$

is equal to R_{TM}.

The significance of this is that R is defined for arbitrary vector bundles, without reference to any structures of bracket type. Thus the Tulczyjew isomorphism could be intrinsically defined in terms of the symplectic structure on T^*M and the map R. (The definition in [**12**] uses the canonical involution on T^2M and the pairing which induces (3); see the example at the end of §3 below.)

Before we move on to consider the role of bracket structures in this context, it is interesting to observe that Theorem 1 is forced upon us by the simple pursuit of diagrams. Take an arbitrary double vector bundle and apply the tangent functor to get the triple structure in Figure 1(a). (The diagrams should be viewed as coming out of the page to the right.) Now dualize the bundle $TD \longrightarrow D$. The rear square is precisely of the form (1), but when we dualize D we do so over B and so the dual is the horizontal dual. Likewise the left side face is of the form (1) but when we dualize D we do so over A, and so obtain the vertical dual. Now it is easy to guess that the eighth vertex must be C^* and one can then verify that Figure 1(b) is indeed a triple vector bundle. Using the isomorphisms R for the left and rear sides, we know that $T^*D \cong T^*(D^{*V})$ and $T^*D \cong T^*(D^{*H})$ as double vector bundles and



(a) (b)

FIGURE 1.

we are forced to the conclusion that the top face must also be of the form (2), and that D^{*V} and D^{*H} must be duals.

3

Bracket structures arise naturally from the infinitesimalization of Lie groupoids. To proceed further, we thus need to consider general double Lie groupoids as in [**5**].

$$\begin{array}{ccc} S & \Longrightarrow & V \\ \Downarrow & & \Downarrow \\ H & \Longrightarrow & M \end{array} \tag{5}$$

Applying the Lie functor to the vertical structures gives the diagram on the left below [**5**]. That the Lie algebroid $A_V S$ of the vertical structure inherits a groupoid structure with base AV follows directly by observing that the Lie functor — like the tangent functor — preserves diagrams and pullbacks. The horizontal groupoid structures and the vertical Lie algebroid structures are compatible in the same way as in the definition of a double vector bundle above. Now applying the Lie functor a second time, to the horizontal structures, gives the diagram on the right [**7**]. This is a double Lie algebroid, but for the moment we will be concerned with it as a double vector bundle.

$$\begin{array}{ccc} A_V S & \Longrightarrow & AV \\ \downarrow & & \downarrow \\ H & \Longrightarrow & M \end{array} \qquad \begin{array}{ccc} A_H(A_V S) & \longrightarrow & AV \\ \downarrow & & \downarrow \\ AH & \longrightarrow & M \end{array} \tag{6}$$

We could equally well apply the Lie functor first horizontally and then vertically, obtaining the following structures.

$$(7)\qquad \begin{array}{ccc} A_HS & \longrightarrow & V \\ \Downarrow & & \Downarrow \\ AH & \longrightarrow & M \end{array} \qquad\qquad \begin{array}{ccc} A_V(A_HS) & \longrightarrow & AV \\ \downarrow & & \downarrow \\ AH & \longrightarrow & M \end{array}$$

The concept of core extends to double groupoids [**2**], [**5**] and in the case of S above yields a Lie groupoid K on base M, the Lie algebroid of which is the core of the two double vector bundles $A_V(A_HS)$ and $A_H(A_VS)$.

The Lie functor is a kind of generalized tangent functor, and both $A_V(A_HS)$ and $A_H(A_VS)$ are submanifolds of T^2S. Further, the canonical involution J carries $A_V(A_HS)$ onto $A_H(A_VS)$ [**7**]. Denote this restricted map by j_S. This is an isomorphism of double vector bundles, as in Figure 2.

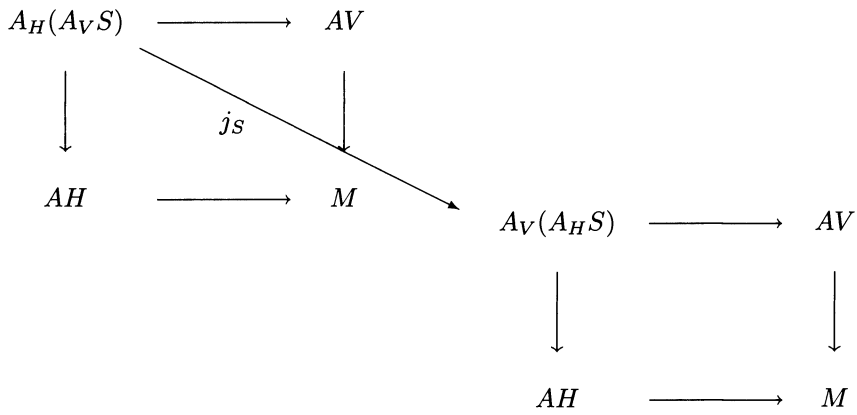


FIGURE 2.

Dualizing j_S over AH we obtain a map

$$j^{*V}\colon A_V^*(A_HS) \longrightarrow A_H^\bullet(A_VS).$$

The structure on the domain is the ordinary dual of the Lie algebroid of a Lie groupoid. On the target, however, the solidus indicates that we have taken the dual of a prolongation structure. As with the case

$$\begin{pmatrix} TE \\ \downarrow \\ TM \end{pmatrix}^* \cong \begin{pmatrix} T(E^*) \\ \downarrow \\ TM \end{pmatrix}$$

in (3), we can show that

$$\begin{pmatrix} A_H(A_V S) \\ \downarrow \\ AH \end{pmatrix}^* \cong \begin{pmatrix} A(A_V^* S) \\ \downarrow \\ AH \end{pmatrix}$$

where

$$\begin{array}{ccc} A_V^* S & \Longrightarrow & A^* K \\ \downarrow & & \downarrow \\ H & \Longrightarrow & M \end{array} \tag{8}$$

is a structure obtained by applying to $A_V S$ an extended form of the duality for double vector bundles (see [**11**] or [**6**]).

Denote the isomorphism $A(A_V^* S) \longrightarrow A_H^\bullet(A_V S)$ by I_V. We then define a generalization of the Tulczyjew isomorphism by

$$\Theta = I_V^{-1} \circ j^{*V} \colon A_V^*(A_H S) \longrightarrow A(A_V^* S)$$

(In [**6**] the inverse of this map is denoted j'^V.)

Example: To recover the classical case, consider the double groupoid

$$S = \begin{array}{ccc} M^4 & \Longrightarrow & M^2 \\ \Downarrow\Downarrow & & \Downarrow\Downarrow \\ M^2 & \Longrightarrow & M \end{array}$$

in which elements are quadruples of points, regarded as the corners of empty squares. Applying the Lie functor vertically, we obtain

$$A_V S = \begin{array}{ccc} TM \times TM & \Longrightarrow & TM \\ \downarrow & & \downarrow \\ M \times M & \Longrightarrow & M \end{array}$$

Here $TM \times TM$ is both the tangent Lie algebroid of $M \times M$ and the pair groupoid on base TM. The dualization process then produces

$$A_V^* S = \begin{array}{ccc} T^*M \times T^*M & \Longrightarrow & T^*M \\ \downarrow & & \downarrow \\ M \times M & \Longrightarrow & M \end{array}$$

in which again the horizontal groupoid structure is a pair groupoid. Applying the Lie functor to this horizontally we obtain

$$A(A_V^* S) = \begin{array}{ccc} T(T^*M) & \longrightarrow & T^*M \\ \downarrow & & \downarrow \\ TM & \longrightarrow & M \end{array}$$

The reader can similarly check that $A_V^*(A_H S) = T^*(TM)$. In this case I_V is precisely the isomorphism induced by the tangent pairing (3) of T^2M and $T(T^*M)$, and the definition of Θ in this case thus reduces exactly to that in [**12**].

4

Finally we consider a general form of the map $T^*(T^*M) \longrightarrow T(T^*M)$ obtained from the symplectic structure on a cotangent bundle.

Theorem 2: [**6**] Given any double Lie groupoid (5) with core groupoid $K \Longrightarrow M$, there is a structure of double Lie groupoid on T^*S which makes it a symplectic double groupoid

$$\begin{array}{ccc} T^*S & \Longrightarrow & A^*_V S \\ \Downarrow & & \Downarrow \\ A^*_H S & \Longrightarrow & A^*K \end{array}$$

and which induces on $A^*_V S$, $A^*_H S$ and A^*K their standard Poisson structures. In particular, $A^*_H S \Longrightarrow A^*K$ and $A^*_V S \Longrightarrow A^*K$ are Poisson groupoids in duality.

The statement that $A^*_H S \Longrightarrow A^*K$ and $A^*_V S \Longrightarrow A^*K$ are Poisson groupoids in duality means that the Lie algebroid of each (as a Lie groupoid) is $(\pm)$–isomorphic to the Lie algebroid dual of the other (with its Lie algebroid structure inherited from the Poisson structure) [**13**]. Thus Theorem 2 asserts the existence of Lie algebroid isomorphisms

$$\mathcal{D}_H \colon A^*(A^*_H S) \longrightarrow A(A^*_V S), \qquad \mathcal{D}_V \colon A^*(A^*_V S) \longrightarrow A(A^*_H S).$$

Following through the Example at the end of §3, the reader can verify that these generalize the map $\pi^{\#}_{d\theta}$. Whereas $\pi^{\#}_{d\theta}$ arises from the symplectic structure on T^*M, the maps $\mathcal{D}_H$ and $\mathcal{D}_V$ arise from the Poisson structures on the duals of Lie algebroids.

In order to express the relationship between these maps $\mathcal{D}$ and the Tulczyjew diffeomorphism in the general case, we need one more ingredient. Consider the structure $A_H S$ from (7). Applying the duality process mentioned in connection with (8), we obtain the structure on the left below. Applying the Lie functor vertically we obtain the double vector bundle on the right below.

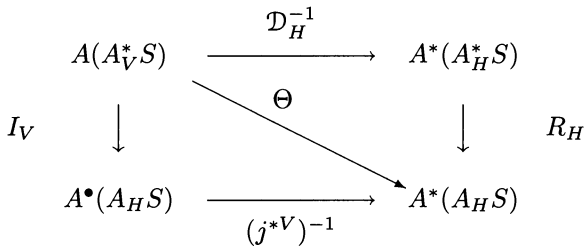
(9)
$$\begin{array}{ccc} A^*_H S & \longrightarrow & V \\ \Downarrow & & \Downarrow \\ A^*K & \longrightarrow & M \end{array} \qquad\qquad \begin{array}{ccc} A(A^*_H S) & \longrightarrow & AV \\ \downarrow & & \downarrow \\ A^*K & \longrightarrow & M \end{array}$$

Now consider the vertical dual of the double vector bundle $A_V(A_H S)$ in (7). By an extension of the methods used in §2, one can define a non–degenerate pairing of the double vector bundles $A(A^*_H S)$ and $A^*_V(A_H S)$which induces an isomorphism of double vector bundles

$$R_H \colon A^*_V(A^*_H S) \longrightarrow A^*(A_H S).$$

In the case where S is actually a double vector bundle, this reduces to the R of §2.

Theorem 3:[**6**] For a double Lie groupoid S as in (5), the following diagram of diffeomorphisms commutes.



In particular, it follows that Θ is a Poisson diffeomorphism, since both $\mathcal{D}_H$ and R_H are anti–Poisson [**6**].

In conclusion, it is worthwhile noticing the different degrees of development required by the various canonical diffeomorphims which we have discussed. The map $R_A\colon T^*(A^*) \longrightarrow T^*(A)$ is the simplest, requiring only a single ordinary vector bundle. The Tulczyjew diffeomorphism requires the notion of double Lie groupoid, and of its associated double Lie algebroid, but involves only the simpler aspects of that construction — a dualization of the map j as a morphism of double vector bundles and the prolongation of a pairing which exists on the level of the intermediate structures A_VS. The maps $\mathcal{D}_V$ and $\mathcal{D}_H$, on the other hand, require a detailed consideration of the structure of the symplectic double groupoid associated with a double Lie groupoid. As was shown in [**6**] and [**8**], this structure is the key to the general notion of double Lie algebroid.

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Department of Pure Mathematics, University of Sheffield, Sheffield, S3 7RH, United Kingdom

E-mail address: K.Mackenzie@sheffield.ac.uk

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Laplacians in odd symplectic geometry

Hovhannes M. Khudaverdian

ABSTRACT. We consider odd Laplace operators arising in odd symplectic geometry. Approach based on semidensities (densities of weight 1/2) is developed. The role of semidensities in the Batalin–Vilkovisky formalism is explained. In particular, we study the relations between semidensities on an odd symplectic supermanifold and differential forms on a purely even Lagrangian submanifold. We establish a criterion of "normality" of a volume form on an odd symplectic supermanifold in terms of the canonical odd Laplacian acting on semidensities.

1. Symplectic and Poisson structures

A symplectic structure on a manifold M is defined by a non-degenerate closed two-form ω. In a vicinity of an arbitrary point one can consider coordinates $(x^1, \ldots, x^{2n})$ such that $\omega = \sum_{i=1}^{n} dx^i dx^{i+n}$. Such coordinates are called Darboux coordinates. To a symplectic structure corresponds a non-degenerate Poisson structure $\{\ ,\ \}$. In Darboux coordinates $\{x^i, x^j\} = 0$ if $|i-j| \neq n$ and $\{x^i, x^{i+n}\} = -\{x^{i+n}, x^i\} = 1$. The condition of closedness of the two-form ω corresponds to the Jacobi identity $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ for the Poisson bracket. If a symplectic or Poisson structure is given, then every function f defines a vector field (the Hamiltonian vector field) $\mathbf{D}_f$ such that $\mathbf{D}_f g = \{f, g\} = -\omega(\mathbf{D}_f, \mathbf{D}_g)$.

A Poisson structure can be defined independently of a symplectic structure (see below). In general it can be degenerate, i.e., there exist non-constant functions f such that $\mathbf{D}_f = 0$. In the case when a Poisson structure is non-degenerate (corresponds to a symplectic structure), the map from T^*M to TM defined by the relation $f \mapsto \mathbf{D}_f$ is an isomorphism.

One can straightforwardly generalize these constructions to the supercase and consider symplectic and Poisson structures (even or odd) on supermanifolds. An even (odd) symplectic structure on a supermanifold is defined by an even (odd) non-degenerate closed two-form. In the same way as the existence of a symplectic structure on an ordinary manifold implies that the manifold is even-dimensional (by the non-degeneracy condition for the form ω), the existence of an even or odd

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On leave of absence from G. S. Sahakyan Department of Theoretical Physics, Yerevan State University, A. Manoukian St., 375049 Yerevan, Armenia.

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symplectic structure on a supermanifold implies that the dimension of the supermanifold is equal either to $(2p.q)$ for an even structure or to $(m.m)$ for an odd structure. Darboux coordinates exist in both cases. For an even structure, the two-form in Darboux coordinates $z^A = (x^1, \dots, x^{2p}; \theta_1, \dots, \theta_q)$ has the form $\sum_{i=1}^{p} dx^i dx^{p+i} + \sum_{a=1}^{q} \varepsilon_a d\theta_a d\theta_a$, where $\varepsilon_a = \pm 1$. For an odd structure, the two-form in Darboux coordinates $z^A = (x^1, \dots, x^m; \theta_1, \dots, \theta_m)$ has the form $\sum_{i=1}^{m} dx^i d\theta_i$.

The non-degenerate odd Poisson bracket corresponding to an odd symplectic structure has the following appearance in Darboux coordinates: $\{x^i, x^j\} = 0$, $\{\theta_i, \theta_j\} = 0$ for all i, j and $\{x^i, \theta_j\} = -\{\theta_j, x^i\} = \delta^i_j$. Thus for arbitrary two functions f, g

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial \theta_i} + (-1)^{p(f)} \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial x^i} \right), \tag{1.1}$$

where we denote by $p(f)$ the parity of a function f (e.g., $p(x^i) = 0$, $p(\theta_j) = 1$). Similarly one can write down the formulae for the non-degenerate even Poisson structure corresponding to an even symplectic structure.

A Poisson structure (odd or even) can be defined on a supermanifold independently of a symplectic structure as a bilinear operation on functions (bracket) satisfying the following relations taken as axioms:

$$p(\{f, g\}) = p(f) + p(g) + \varepsilon, \tag{1.2}$$

$$\{f, g\} = -\{g, f\}(-1)^{(p(f)+\varepsilon)(p(g)+\varepsilon)}, \tag{1.3}$$

$$\{f, gh\} = \{f, g\}h + \{f, h\}g(-1)^{p(g)p(h)} \quad \text{(Leibniz rule)}, \tag{1.4}$$

$$\{f, \{g, h\}\}(-1)^{(p(f)+\varepsilon)(p(h)+\varepsilon)} + \text{cycl.} = 0 \quad \text{(Jacobi identity)}, \tag{1.5}$$

where ε is the parity of the bracket ($\varepsilon = 0$ for an even Poisson structure and $\varepsilon = 1$ for an odd one). The correspondence between functions and Hamiltonian vector fields is defined in the same way as on ordinary manifolds: $\mathbf{D}_f g = \{f, g\}$. Notice a possible parity shift: $p(\mathbf{D}_f) = p(f) + \varepsilon$. Every Hamiltonian vector field $\mathbf{D}_f$ defines an infinitesimal transformation preserving the Poisson structure (and the corresponding symplectic structure in the case of a non-degenerate Poisson bracket).

Notice that even or odd Poisson structures on an arbitrary supermanifold can be obtained as "derived" brackets from the canonical symplectic structure on the cotangent bundle, in the following way.

Let M be a supermanifold and T^*M be its cotangent bundle. By changing parity of coordinates in the fibres of T^*M we arrive at the supermanifold ΠT^*M. If z^A are arbitrary coordinates on the supermanifold M, then we denote by (z^A, p_B) the corresponding coordinates on the supermanifold T^*M and by (z^A, z_{*B}) the corresponding coordinates on ΠT^*M: $p(z^A) = p(p_A) = p(z_{*A}) + 1$. If $(z^{A'})$ are another coordinates on M, $z^A = z^A(z')$, then the coordinates z_{*A} transform in the same way as the coordinates p_A (and as the partial derivatives $\partial/\partial z^A$):

$$p_{A'} = \frac{\partial z^B(z')}{\partial z^{A'}} p_B \quad \text{and} \quad z_{*A'} = \frac{\partial z^B(z')}{\partial z^{A'}} z_{*B}. \tag{1.6}$$

One can consider the canonical non-degenerate even Poisson structure $\{\ ,\ \}_0$ (the canonical even symplectic structure) on T^*M defined by the relations $\{z^A, z^B\}_0 = \{p_C, p_D\}_0 = 0$, $\{z^A, p_B\}_0 = \delta^A_B$, and, respectively, the canonical non-degenerate

odd Poisson structure $\{\ ,\ \}_1$ (the canonical odd symplectic structure) on ΠT^*M defined by the relations $\{z^A, z^B\}_0 = \{z_{*C}, z_{*D}\}_0 = 0$, $\{z^A, z_{*B}\}_0 = \delta^A_B$.

Now consider Hamiltonians on T^*M or on ΠT^*M that are quadratic in coordinates of the fibres. An arbitrary odd quadratic Hamiltonian on T^*M (an arbitrary even quadratic Hamiltonian on ΠT^*M):

$$\mathfrak{S}(z,p) = \mathfrak{S}^{AB} p_A p_B \ \ (p(\mathfrak{S}) = 1) \quad \text{or} \quad \mathfrak{S}(z, z_*) = \mathfrak{S}^{AB} z_{*A} z_{*B} \ \ (p(\mathfrak{S}) = 0), \tag{1.7}$$

satisfying the condition that the canonical Poisson bracket of this Hamiltonian with itself vanishes:

$$\{\mathfrak{S}, \mathfrak{S}\}_0 = 0 \quad \text{or} \quad \{\mathfrak{S}, \mathfrak{S}\}_1 = 0 \tag{1.8}$$

defines an odd Poisson structure (an even Poisson structure) on M by the formula

$$\{f, g\}^{\mathfrak{S}}_{\varepsilon+1} = \{f, \{\mathfrak{S}, g\}_\varepsilon\}_\varepsilon\,. \tag{1.9}$$

The Hamiltonian $\mathfrak{S}$ which generates an odd (even) Poisson structure on M via the canonical even (odd) Poisson structure on T^*M (ΠT^*M) can be called the master Hamiltonian. The bracket (1.9) is a "derived bracket". The Jacobi identity for it is equivalent to the vanishing of the canonical Poisson bracket for the master Hamiltonian. One can see that an arbitrary Poisson structure on a supermanifold can be obtained as a derived bracket.

What happens if we change the parity of the master Hamiltonian in (1.9)? The answer is the following. If $\mathfrak{S}$ is an even quadratic Hamiltonian on T^*M (an odd quadratic Hamiltonian on ΠT^*M), then the condition of vanishing of the canonical even Poisson bracket $\{\ ,\ \}_0$ (the canonical odd Poisson bracket $\{\ ,\ \}_1$) becomes empty (it is obeyed automatically) and the relation (1.9) defines an even Riemannian metric (an odd Riemannian metric) on M.

Formally, odd symplectic (and odd Poisson) geometry is a generalization of symplectic (Poisson) geometry to the supercase. However, there are unexpected analogies between the constructions in odd symplectic geometry and in Riemannian geometry (see [**10**] and later below). The construction of derived brackets could explain close relations between odd Poisson structures in supermathematics and the Riemannian geometry (see [**10**]).

The construction of the derived bracket (without the name) and the elaboration of the unified viewpoint for different geometries in terms of derived brackets are due to T. Voronov [**17**]. Derived brackets (under this name) were independently introduced and studied in [**11**]. It has to be noted that in the physical literature the relations of the type (1.9) for brackets of different parity were considered in [**12**] and [**13**], where they were used for obtaining derived brackets on Lagrangian surfaces. This approach was considered later in [**5**] and essentially developed in [**4**], where constructions involving generalized "higher order" even and odd Poisson brackets appeared.

In what follows we consider second order differential operators on an odd symplectic supermanifold and study their geometric properties. Some of our constructions can be automatically considered in the case of a general odd Poisson structure.

2. Odd Laplacians on functions

2.1. Definition and properties. In ordinary symplectic geometry the symplectomorphisms of M^{2n} (the diffeomorphisms preserving the two-form ω) preserve the volume form $\boldsymbol{\rho}_\omega = \omega^n$ (Liouville's theorem). What is the situation in the

supercase? In spite of the fact that differential forms are not objects of integration on supermanifolds (see details in [**16**]), the Liouville theorem still holds in even symplectic geometry. One can see that the coordinate volume form $\boldsymbol{\rho} = \mathcal{D}z$ in Darboux coordinates defines a global volume form that is preserved under symplectomorphisms. The situation is drastically different for an odd symplectic structure. Let $z^A = (x^1, \dots, x^n, \theta_1, \dots, \theta_n)$ be Darboux coordinates. Consider, for example, the transformation $x^1 \mapsto 2x^1$, $\theta_1 \mapsto \frac{1}{2}\theta_1$ to another Darboux coordinates. The Berezinian of this transformation is equal to 4, hence the coordinate volume $\mathcal{D}z$ form is not preserved. One can prove (see below) that on an odd symplectic supermanifold there is no volume form invariant w.r.t. all symplectomorphisms.

Let $\boldsymbol{\rho}$ be an arbitrary volume form on an odd symplectic supermanifold. Consider the linear differential operator $\Delta_{\boldsymbol{\rho}}$ on functions such that its action on a function f is equal (up to a coefficient) to the divergence $\operatorname{div}_{\boldsymbol{\rho}}\mathbf{D}_f$ of the Hamiltonian vector field $\mathbf{D}_f$ w.r.t. to the volume form $\boldsymbol{\rho}$:

$$\Delta_{\boldsymbol{\rho}} f := \frac{1}{2}(-1)^{p(f)}\operatorname{div}_{\boldsymbol{\rho}}\mathbf{D}_f = \frac{1}{2}(-1)^{p(f)}\mathcal{L}_{\mathbf{D}_f}\log\boldsymbol{\rho} = \frac{1}{2}(-1)^{p(f)}\frac{\mathcal{L}_{\mathbf{D}_f}\boldsymbol{\rho}}{\boldsymbol{\rho}}. \tag{2.1}$$

Let $(x^1, \dots, x^n; \theta_1, \dots, \theta_n)$ be Darboux coordinates and let $\boldsymbol{\rho} = \rho(x,\theta)\mathcal{D}(x,\theta)$ in these coordinates. It follows from (1.1) that

$$\mathbf{D}_f = \frac{\partial f}{\partial x^i}\frac{\partial}{\partial\theta_i} + (-1)^{p(f)}\frac{\partial f}{\partial\theta_i}\frac{\partial}{\partial x^i}.$$

Hence

$$\Delta_{\boldsymbol{\rho}} f = \Delta_0 f + \frac{1}{2}\{\log\rho, f\}, \tag{2.2}$$

where

$$\Delta_0 f = \frac{\partial^2 f}{\partial x^i \partial\theta_i}. \tag{2.3}$$

We come to the *odd Laplacian on functions*, a second order differential operator depending on the volume form.

Notice that (2.1) defines the operator $\Delta_{\boldsymbol{\rho}}$ in terms of the Poisson bracket. This expression defines a linear operator on functions for an arbitrary Poisson structure, even or odd. It is an operator of the second order for an odd Poisson structure and an operator of the first order for an even Poisson structure. In the even case this operator of the first order is a Poisson vector field (the divergence of the Poisson bivector) specifying the so-called Weinstein's modular class of an even Poisson manifold [**18**].

If $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ are two volume forms, $\boldsymbol{\rho}' = g\boldsymbol{\rho}$, then

$$\operatorname{div}_{\boldsymbol{\rho}'} f - \operatorname{div}_{\boldsymbol{\rho}} f = \frac{\mathcal{L}_{\mathbf{D}_f} g}{g} = \{f, \log g\}. \tag{2.4}$$

In particular, the existence of a "canonical" volume form $\boldsymbol{\rho}_\omega$ preserved under all symplectomorphisms would imply that the operator (2.1) is a first order differential operator given by the r.h.s. of (2.4), because (2.1) would evidently vanish for $\boldsymbol{\rho} = \boldsymbol{\rho}_\omega$. On the other hand, by (2.2) and (2.3), Δ_ρ is a second order differential operator. Thus we have proved that *on an odd symplectic supermanifold there is no canonical volume form.*

This should be compared with the even Poisson situation where Weinstein's modular class (see above) is the obstruction to the existence of a volume form invariant under all Hamiltonian flows, and it vanishes in the (even) symplectic case. We see that the situation with the odd bracket (symplectic or not) is more complicated.

Now consider the properties of the odd Laplacian Δ_ρ.

One can see that for an arbitrary odd Poisson supermanifold the Leibniz rule for the second derivatives takes the following form for the odd Laplacian:

$$\Delta_\rho(f \cdot g) = \Delta_\rho f \cdot g + (-1)^{p(f)}\{f, g\} + (-1)^{p(f)} f \cdot \Delta_\rho g\,. \tag{2.5}$$

In other words, the operator Δ_ρ generates the Poisson structure.

We have already mentioned that (2.5) has a straightforward analogue in Riemannian geometry: $\Delta(fg) = g\Delta f + \langle df, dg\rangle + f\Delta g$, where $\langle\ ,\ \rangle$ is the scalar product given by the Riemannian metric and Δ is the Beltrami–Laplace operator corresponding to the metric (see details in [**10**]).

Another very important property of the odd Laplacian (on an arbitrary odd Poisson supermanifold) is that it preserves the Poisson bracket:

$$\Delta_\rho\{f, g\} = \{\Delta_\rho f, g\} + (-1)^{p(f)+1}\{f, \Delta_\rho g\}. \tag{2.6}$$

Now let us return to the canonical odd symplectic structure on ΠT^*M (see the first section). We consider the case when M is a usual n-dimensional manifold. The base manifold M is a Lagrangian $(n.0)$-dimensional surface in the $(n.n)$-dimensional odd symplectic supermanifold ΠT^*M. This example can be considered as the basic example of an odd symplectic supermanifold [1].

Functions on ΠT^*M encode multivector fields on M:

$$f(x, \theta) = f(x) + f^i(x)\theta_i + f^{ik}(x)\theta_i\theta_k + \ldots \tag{2.7}$$
$$\leftrightarrow T = f(x) + f^i(x)\partial_i + f^{ik}(x)\partial_i \wedge \partial_k + \ldots$$

The odd Poisson bracket of functions on ΠT^*M corresponds to the Schouten bracket ("skew-symmetric concomitant") of multivector fields.

To every diffeomorphism of M naturally corresponds the induced symplectomorphism of ΠT^*M, but in general one can consider symplectomorphisms that do not correspond to diffeomorphisms of the base and destroy the cotangent bundle structure (see the next section).

Let us now analyze the meaning of an odd Laplacian (2.1) on ΠT^*M. Let (x^i) be arbitrary coordinates on M and (x^i, θ_j) the corresponding Darboux coordinates on ΠT^*M. Let $\sigma = \mathcal{D}(x)$ and $\rho = \mathcal{D}(x, \theta)$ be the coordinate volume forms on M and ΠT^*M respectively. Then the odd Laplacian Δ_ρ on ΠT^*M is given by the formula (2.3). It is obvious from (2.3) that in this case the action of the operator Δ_ρ on functions on ΠT^*M corresponds to the divergence of multivector fields on M with respect to the coordinate volume form σ. Every volume form on M has a local appearance as a coordinate volume form (in some local coordinate system). On the other hand, it follows from (1.6) that the determinant of an arbitrary coordinate

[1] For a usual symplectic manifold E and a Lagrangian surface L in it there exists a symplectomorphism between the cotangent bundle T^*L and, in general, only a tubular neighborhood of L in E. The triviality of topology in odd directions allows to identify the cotangent bundle ΠT^*L to a purely even Lagrangian surface in an odd symplectic supermanifold E with the whole supermanifold E if L coincides with the underlying manifold of E (see [**9**] for details).

transformation $x \mapsto x' = x'(x)$ on M is equal to the square root of the Berezinian of the corresponding coordinate transformation $(x, \theta) \mapsto (x', \theta')$:

$$\operatorname{Ber} \frac{\partial(x', \theta')}{\partial(x, \theta)} = \operatorname{Ber} \begin{pmatrix} \frac{\partial x^{i'}}{\partial x^i} & * \\ 0 & \frac{\partial x^i}{\partial x^{i'}} \end{pmatrix} = \left(\det \left(\frac{\partial x^{i'}}{\partial x^i} \right) \right)^2 . \tag{2.8}$$

Hence we come to important conclusions: (a) to every volume form $\boldsymbol{\sigma}$ on M corresponds a volume form $\boldsymbol{\rho} = \boldsymbol{\sigma}^2$ on ΠT^*M; (b) $\operatorname{div}_{\boldsymbol{\sigma}} T = \Delta_{\boldsymbol{\sigma}^2} f$, where we identify multivector fields on M with functions on ΠT^*M by (2.7); (c) for a volume form $\boldsymbol{\rho} = \boldsymbol{\sigma}^2$ on ΠT^*M holds the condition

$$\Delta_{\boldsymbol{\rho}}^2 = 0, \tag{2.9}$$

because the square of the divergence operator on multivector fields equals zero.

These relations make a bridge between odd symplectic geometry and classical vector calculus. They are closely related with the geometric meaning of the Batalin–Vilkovisky formalism (see [**8**]).

What can we say about $\Delta_{\boldsymbol{\rho}}^2$ in the general case?

For an arbitrary odd Poisson supermanifold the operator $\Delta_{\boldsymbol{\rho}}^2$ is a Poisson vector field: $\Delta_{\boldsymbol{\rho}}^2(fg) = (\Delta_{\boldsymbol{\rho}}^2 f)g + f(\Delta_{\boldsymbol{\rho}}^2 g)$ and $\Delta_{\boldsymbol{\rho}}^2$ preserves the Poisson bracket. This follows from relations (2.5), (2.6). Under a change of volume form $\boldsymbol{\rho} \mapsto \boldsymbol{\rho}' = g\boldsymbol{\rho}$ this Poisson vector field changes by a Hamiltonian vector field:

$$\Delta_{\boldsymbol{\rho}'}^2 - \Delta_{\boldsymbol{\rho}}^2 = \mathbf{D}_{H(\boldsymbol{\rho}', \boldsymbol{\rho})} \quad \text{where} \quad H(\boldsymbol{\rho}', \boldsymbol{\rho}) = \frac{1}{\sqrt{g}} \Delta_{\boldsymbol{\rho}} \sqrt{g} \, . \tag{2.10}$$

This relation leads to a non-trivial groupoid structure [**10**].

For an odd symplectic supermanifold one can always pick a volume form $\boldsymbol{\rho}$ such that $\Delta_{\boldsymbol{\rho}}^2 = 0$, namely, as we have shown above, one can identify the symplectic supermanifold with ΠT^*L for an purely even Lagrangian surface L by a suitable symplectomorphism and then choose $\boldsymbol{\rho} = \boldsymbol{\sigma}^2$, yielding (2.9). Hence, it follows from (2.10) that for odd Laplacians on a symplectic supermanifold the operator $\Delta_{\boldsymbol{\rho}}^2$ is always a Hamiltonian vector field.

DEFINITION 2.1. A volume form $\boldsymbol{\rho}$ on an odd symplectic supermanifold E is called *normal* if in a vicinity of an arbitrary point there exist Darboux coordinates (x, θ) such that $\boldsymbol{\rho}$ is the coordinate volume form in these Darboux coordinates: $\boldsymbol{\rho} = \mathcal{D}(x, \theta)$.

The volume form $\boldsymbol{\rho} = \boldsymbol{\sigma}^2$ on ΠT^*M in (2.9) is a normal volume form by definition. If a volume form $\boldsymbol{\rho}$ is normal, then $\Delta_{\boldsymbol{\rho}}^2 = 0$, by the definition of the odd Laplacian (2.2). Does the Hamiltonian field $\Delta_{\boldsymbol{\rho}}^2$ has to be equal to zero for every volume form $\boldsymbol{\rho}$? As it follows from (2.10), the answer is, generally, no. Does the condition $\Delta_{\boldsymbol{\rho}}^2 = 0$ implies that $\boldsymbol{\rho}$ is a normal volume form? We will give a detailed analysis in the next sections, where we will study the canonical odd Laplacian acting on semidensities. Now two words about where odd Laplacians come from.

2.2. Where an odd Laplacian comes from. Odd Laplacians have appeared in mathematical physics for the first time around 1981 in the pioneer works by Batalin and Vilkovisky [**1, 2, 3**] for the purpose of constructing a Lagrangian version of the BRST quantization (the "BV-formalism"). Batalin and Vilkovisky introduced an odd Laplacian acting on functions on an odd symplectic supermanifold as $\Delta_0 = \frac{\partial^2}{\partial x^i \partial \theta_i}$, where (x^i, θ_i) are some Darboux coordinates. (The invariant

definition (2.1) of an odd Laplacian Δ_ρ depending on a volume form $\boldsymbol{\rho}$ was given later in [**6**].) The following very important properties of this operator were fixed in their works (see [**3**]). If (x', θ') are another Darboux coordinates, and Δ_0 and Δ_0' denote the odd Laplacians (2.3) in the Darboux coordinates (x, θ) and (x', θ') respectively, then

$$\Delta_0 = \Delta_0' + \frac{1}{2}\{\log \operatorname{Ber} \frac{\partial(x', \theta')}{\partial(x, \theta)}, \ \} \tag{2.11}$$

and

$$\Delta_0 \left(\operatorname{Ber} \frac{\partial(x', \theta')}{\partial(x, \theta)} \right)^{1/2} = 0 \quad (\text{"Batalin–Vilkovisky identity"}). \tag{2.12}$$

The property (2.11) of the operator Δ_0 is closely related with the invariant expression (2.2) for Δ_ρ. The second property (2.12), which was stated in [**3**] (in a non-explicit form it also appeared in [**15**]), is highly non-trivial. This identity is deeply related with the existence of canonical odd Laplacian on semidensities (see the next section).

The operator Δ_0 was introduced in [**1**] for the purpose of formulating the so-called Batalin–Vilkovisky quantum master equation, which is the equation

$$\Delta_0 \sqrt{f(x, \theta)} = 0 \tag{2.13}$$

on the function $f(x, \theta) = \exp \frac{i\mathcal{S}(x,\theta)}{\hbar}$, where $\boldsymbol{\rho} = \mathcal{D}(x, \theta)$ is a coordinate volume form in the space of fields and antifields ((x, θ) are Darboux coordinates). The master action $\mathcal{S}(x, \theta)$ defines a measure element on Lagrangian surfaces corresponding to gauge choice. This measure is gauge invariant if the master equation is satisfied (see [**1, 2, 3**]). Using the identity $\Delta_0 \exp g = (\Delta_0 g + 1/2\{g, g\}) \exp g$, we can rewrite the quantum master equation (2.13) as $-4\hbar\Delta_0\mathcal{S} + \{\mathcal{S}, \mathcal{S}\} = 0$ and taking $\hbar \to 0$ we arrive at the Batalin–Vilkovisky classical master equation: $\{\mathcal{S}, \mathcal{S}\} = 0$. The geometrical meaning of the master equation was studied in [**7, 8**] and most notably by A. S. Schwarz in [**14**]. In particular, the following result was obtained. Suppose $\boldsymbol{\rho}$ is a normal volume form. Then there are implications:

$$\text{the volume form } \boldsymbol{\rho}' = f(x, \theta)\boldsymbol{\rho} \text{ is normal} \ \Rightarrow \ \Delta_{\boldsymbol{\rho}} \sqrt{f} = 0 \ \Rightarrow \ \Delta^2_{\boldsymbol{\rho}'} = 0\,. \tag{2.14}$$

In this statement the master equation is not formulated invariantly, but it stands between two invariant conditions. The exact statement about the relations between the three conditions in (2.14) will be formulated in the next section in the language of semidensities.

3. Canonical odd Laplacian on semidensities

3.1. Definition of the canonical Laplacian. A density of weight t on a supermanifold is a function of local coordinates such that under a change of variables it is multiplied by the t-th power of the Berezinian of transformation. We will consider semidensities (densities of weight $t = \frac{1}{2}$) on an odd symplectic supermanifold.

First of all, let us consider again the supermanifold ΠT^*M for a usual manifold M, with the canonical symplectic structure. Recall that functions on ΠT^*M encode multivector fields on M (see (2.7)). Our claim [**9**] is that *semidensities on* ΠT^*M *encode differential forms on* M.

Indeed, let $(x^1, \ldots, x^n)$ be arbitrary local coordinates on the manifold M. Let $(x^1, \ldots, x^n\,; \theta_1, \ldots, \theta_n)$ be the corresponding Darboux coordinates on ΠT^*M. In

the same way as multivector fields on M can be identified with functions on ΠT^*M, differential forms on M can be identified with functions on the supermanifold ΠTM obtained from the tangent bundle TM by changing parity of coordinates in the fibres. If $(x^1,\dots,x^n,\xi^1,\dots,\xi^n)$ are the coordinates on ΠTM corresponding to coordinates $(x^1,\dots,x^n)$ on M ($p(\xi^k)=1$), then a differential form $\omega_0(x)+dx^i\omega_i(x)+\dots$ can be identified with the function $\omega(x,\xi)=\omega_0(x)+\xi^i\omega_i(x)+\dots$ on ΠTM.

Recall that given a volume form $\boldsymbol{\sigma}=\sigma(x)\mathcal{D}x=\sigma(x)dx^1\dots dx^n$ on M the Hodge operator transforms a multivector field on M corresponding to the function $f(x,\theta)$ on ΠT^*M to the differential form on M corresponding to the function $\omega(x,\xi)$ on ΠTM, where

$$\omega(x,\xi)=\int \exp(i\xi^i\theta_i)f(x,\theta)\,\sigma(x)\mathcal{D}\theta$$

(a "Fourier transform"). (In a conventional language it is a contraction of a top order form with a multivector, but the language of integrals is more flexible.) It follows that without a volume form, the Hodge operator acts on multivector densities of weight $t=1$ on M transforming them into differential forms on M, and vice versa.

On the other hand, from (2.8) it follows that under canonical transformations on ΠT^*M induced by changes of coordinates of M a density on M transforms as a semidensity on ΠT^*M.

Hence we come to a 1-1-correspondence between differential forms on M (functions on ΠTM) and semidensities on ΠT^*M, as follows[2]:

$$\omega(x,\xi)\mapsto \boldsymbol{s}=s(x,\theta)\sqrt{\mathcal{D}(x,\theta)}=\left(\int \exp(-i\xi\theta)\omega(x,\xi)\mathcal{D}\xi\right)\sqrt{D(x,\theta)} \tag{3.1}$$

(This map for the first time appeared in [**14**] in a non-explicit way.) For example, if M is a two-dimensional manifold, then $f(x)\mapsto f(x)\theta_1\theta_2\sqrt{\mathcal{D}(x,\theta)}$, $\omega_1dx^1+\omega_2dx^2\mapsto(\omega_1\theta_2-\omega_2\theta_1)\sqrt{\mathcal{D}(x,\theta)}$, $\omega dx^1dx^2\mapsto-\omega\sqrt{\mathcal{D}(x,\theta)}$

The relation (3.1) between forms on M and semidensities on ΠT^*M suggests that there exists a linear operator on semidensities on odd symplectic supermanifolds corresponding to the exterior differential.

DEFINITION 3.1 ([**9**]). Let $\boldsymbol{s}$ be a semidensity on an odd symplectic supermanifold E. We assign to it a semidensity $\Delta\boldsymbol{s}$ by the following formula: if $\boldsymbol{s}=s(x,\theta)\sqrt{\mathcal{D}(x,\theta)}$ in some Darboux coordinates, then in the same coordinates

$$\Delta\boldsymbol{s}:=\big(\Delta_0 s(x,\theta)\big)\sqrt{\mathcal{D}(x,\theta)}=\frac{\partial^2 s}{\partial x^i\partial\theta_i}\sqrt{\mathcal{D}(x,\theta)}\,. \tag{3.2}$$

We call this operator *the canonical Laplacian on semidensities.* (See [**9**] for details.)

In the case of $E=\Pi T^*M$ one can see from (3.1) that this definition gives exactly the de Rham exterior differential:

$$\omega\mapsto\boldsymbol{s}_\omega\quad\Rightarrow\quad\Delta\boldsymbol{s}_\omega=\boldsymbol{s}_{d\omega}\,. \tag{3.3}$$

Of course, this relation is not a proof that the operator given by (3.2) is well-defined for a general odd symplectic supermanifold E, because though one might consider E as ΠT^*M for some manifold M the identification (3.1) fails under symplectomorphisms which are not induced by diffeomorphisms of M.

[2]In these considerations we assume that the manifold M is orientable and an orientation is chosen.

To prove that the canonical operator is well-defined by formula (3.2) one has to prove that the r.h.s. of (3.2) indeed defines a semidensity, i.e., under an arbitrary transformation from Darboux coordinates (x,θ) to another Darboux coordinates (x',θ') we have

$$(\Delta_0 s)\cdot\left(\mathrm{Ber}\frac{\partial(x,\theta)}{\partial(x',\theta')}\right)^{1/2}=\Delta_0'\left(s\cdot\left(\mathrm{Ber}\frac{\partial(x,\theta)}{\partial(x',\theta')}\right)^{1/2}\right), \tag{3.4}$$

where we denote by Δ_0 and Δ_0' the "coordinate" odd Laplacians (2.3) in the Darboux coordinates (x,θ) and (x',θ'), respectively.

Notice (see [**9**] for details) that every transformation from Darboux coordinates (x,θ) to new Darboux coordinates (x',θ') can be represented as the composition of transformations of the following types:

transformations corresponding to $x'=x'(x)$:

$$x'=x'(x),\qquad \theta_{i'}=\frac{\partial x^i}{\partial x^{i'}}\,\theta_i\,, \tag{3.5}$$

transformations identical on the surface $\theta=0$:

$$x'(x,\theta)\big|_{\theta=0}=x\,,\qquad \theta'(x,\theta)\big|_{\theta=0}=\theta\,, \tag{3.6}$$

transformations identical on even coordinates:

$$x'=x,\qquad \theta_{i'}=\theta_i+\alpha_i\quad\text{such that}\quad \partial_i\alpha_j-\partial_j\alpha_i=0\,. \tag{3.7}$$

It is sufficient to check condition (3.4) for transformations (3.5), (3.6) and (3.7) separately. For transformations (3.5) it follows from the identity (2.8). The Berezinian of transformation (3.7) equals 1, hence (3.4) is satisfied. One can show that transformation (3.6) is induced by a Hamiltonian vector field (see [**9**]), hence and it is sufficient to check it infinitesimally.

Infinitesimal transformations are generated by odd functions (Hamiltonians) via the corresponding Hamiltonian vector fields. To an odd Hamiltonian $Q(z)$ corresponds the infinitesimal canonical transformation $\tilde z^A=z^A+\varepsilon\{Q,z^A\}$ generated by the vector field $\mathbf{D}_Q$. The action of it on a semidensity $\boldsymbol{s}$ can be expressed by a "differential" $\delta_Q(s\sqrt{\mathcal{D}z})=\Delta_0 Q\cdot s\sqrt{\mathcal{D}z}-\{Q,s\}\sqrt{\mathcal{D}z}$, because $\delta s=-\varepsilon\{Q,s\}$ and $\delta\mathcal{D}z=\varepsilon\delta\mathrm{Ber}(\partial z/\partial\tilde z)\mathcal{D}z=\varepsilon 2\Delta_0 Q\mathcal{D}z$ for the infinitesimal transformation generated by Q. Using $\Delta_0^2=0$ and equation (2.5), we come to the commutation relation $\Delta_0\delta_Q=\delta_Q\Delta_0$. Thus condition (3.4) is satisfied for infinitesimal transformations.

3.2. Properties of the canonical Laplacian. The canonical Laplacian obviously obeys the condition $\Delta^2=0$.

Let $\boldsymbol{\rho}$ be an arbitrary volume form on an odd symplectic supermanifold. Then it is easy to check using (2.2), (2.5) and (2.6) that the canonical Laplacian on semidensities Δ obeys the following condition:

$$\Delta(f\sqrt{\boldsymbol{\rho}})=(\Delta_{\boldsymbol{\rho}}f)\sqrt{\boldsymbol{\rho}}+(-1)^{p(f)}f\Delta\sqrt{\boldsymbol{\rho}}\,, \tag{3.8}$$

where $\Delta_{\boldsymbol{\rho}}$ is the Laplacian (2.2) on functions. Using (2.4) one can rewrite this relation in the following way, for the semidensity $\boldsymbol{s}=\sqrt{\boldsymbol{\rho}}$:

$$[\Delta,f]\boldsymbol{s}\equiv(\Delta\circ f-(-1)^{p(f)}f\circ\Delta)\boldsymbol{s}=\mathcal{L}_{\mathbf{D}_f}\boldsymbol{s}\,, \tag{3.9}$$

where $\mathcal{L}_{\mathbf{D}_f}$ is the Lie derivative along the Hamiltonian vector field $\mathbf{D}_f$. Notice that (3.9) holds for an arbitrary semidensity $\boldsymbol{s}$, not only for an even non-degenerate semidensity $\boldsymbol{s}=\sqrt{\boldsymbol{\rho}}$ corresponding to a volume form $\boldsymbol{\rho}$. This relation is very

important for the study of Laplacians on semidensities on arbitrary odd Poisson supermanifolds (see [**10**]).

If $\boldsymbol{\rho}$ is an arbitrary volume form, then by applying the canonical Laplacian to the semidensity $\boldsymbol{s} = \sqrt{\boldsymbol{\rho}}$ we can obtain a "derived" function $H = \Delta\sqrt{\boldsymbol{\rho}}/\sqrt{\boldsymbol{\rho}}$. It turns out that the Hamiltonian vector field $\mathbf{D}_H$ corresponding to H is nothing but the vector field $\Delta^2_{\boldsymbol{\rho}}$:

$$\Delta^2_{\boldsymbol{\rho}} f = \left\{ \frac{\Delta\sqrt{\boldsymbol{\rho}}}{\sqrt{\boldsymbol{\rho}}}, f \right\} . \tag{3.10}$$

(Compare this formula with (2.10).)

It is evident that if the form $\boldsymbol{\rho}$ is normal, then by definition

$$\Delta\sqrt{\boldsymbol{\rho}} = 0 . \tag{3.11}$$

This is just an invariant expression for the Batalin–Vilkovisky identity (2.12)

Now let us return to the relation between differential forms on M and semidensities on ΠT^*M, and to its generalization for arbitrary odd symplectic supermanifolds.

We call a semidensity $\boldsymbol{s}$ *closed* or *exact* if $\Delta\boldsymbol{s} = 0$ or $\boldsymbol{s} = \Delta\boldsymbol{r}$ respectively. The condition $\Delta^2 = 0$ for the canonical Laplacian corresponds to $d^2 = 0$ for the exterior differential.

Equations (3.1), (3.3) and (3.9) allow the translation of formulae of vector calculus on M into formulae for semidensities on ΠT^*M (see [**9**]). For example, under the map (3.1) the "interior multiplication" of a differential form ω by a multivector field T transforms into the usual product of the semidensity $\boldsymbol{s}_\omega$ with the function corresponding to the multivector field. Hence, equation (3.9) corresponds to the formula for the Lie derivative of a differential form along a multivector field (a generalization of Cartan's homotopy formula).

Consider in more details the following two constructions which do not appear naturally in classical calculus of forms. (Below we use familiar formal properties of the Fourier transform.)

a) If $a = a_i(x)dx^i$ is a 1-form on M and a semidensity $\boldsymbol{s} = s(x,\theta)\sqrt{\mathcal{D}(x,\theta)}$ on ΠT^*M corresponds to another form ω, then one can see that the semidensity $a_i\frac{\partial s}{\partial\theta_i}\sqrt{D(x,\theta)}$ corresponds to the form $a \wedge \omega$. (Compare with the familiar relation between the differentiation and multiplication by a coordinate for the classical Fourier transform.) Consider the following generalization. Let $a = a_i dx^i$ be a one-form on M with odd coefficients (we have to allow "external odd parameters" for this). For an arbitrary semidensity $\boldsymbol{s} = s(x,\theta)\sqrt{D(x,\theta)}$ consider a new semidensity $\boldsymbol{s}'$, which we denote by $a \lceil \boldsymbol{s}$, given by the formula $\boldsymbol{s}' = a \lceil \boldsymbol{s} := s(x,\theta_i+a_i)\sqrt{D(x,\theta)}$. It is a well-defined operation, because the coefficients a_i have the same transformation law as the variables θ_i. (Notice that the Berezinian of the transformation $(x^i,\theta_i) \mapsto (x^i,\theta_i+a_i)$ equals 1.) Respectively, if the semidensity $\boldsymbol{s}$ corresponds to a differential form $\omega = \sum\omega_k$, then we denote by $a \lceil \omega$ the differential form such that the semidensity $a \lceil \boldsymbol{s}$ corresponds to $a \lceil \omega$. One can see that

$$a \lceil \omega = \sum_{p=0}^{k} \frac{1}{p!} \underbrace{a \wedge \cdots \wedge a}_{p \quad \text{times}} \wedge \omega_{k-p} , \quad (k = 0, \ldots, n) .$$

We obtain an action of the abelian supergroup of differential one-forms "with odd values" (i.e., $\Pi\Omega^1(M)$) in the spaces of semidensities and differential forms (see [**9**]).

b) Let $\omega = \sum \omega_k$ and $\omega' = \sum \omega'_k$ be differential forms on M^n such that their top-degree components ω_n and ω'_n are non-zero, and let the semidensities $\boldsymbol{s}$ and $\boldsymbol{s}'$ correspond to ω and ω' respectively. Then we can define a new form $\tilde{\omega} := \omega * \omega'$ such that the corresponding semidensity is equal to $\sqrt{\boldsymbol{s} \cdot \boldsymbol{s}'}$. The condition $\omega_n \neq 0$, $\omega'_n \neq 0$ for the top-degree components makes the square root operation uniquely defined.

The 1-1 correspondence between forms on M and semidensities on ΠT^*M is defined using the cotangent bundle structure on the odd symplectic supermanifold ΠT^*M. Bearing in mind that every odd symplectic manifold E with an underlying manifold M is symplectomorphic to ΠT^*M (see [**9**] and the footnote in subsection 2.1), let us analyze the relation between semidensities on E and differential forms on M. The map (3.1) is not invariant under arbitrary symplectomorphisms of the total symplectic supermanifold $E = \Pi T^*M$. In other words, if L is an arbitrary $(n.0)$-dimensional Lagrangian surface in E, then the correspondence between semidensities on E and differential forms on L depends on an *identifying symplectomorphism*, i.e. a symplectomorphism $\varphi : \Pi T^*L \to E$ such that $\varphi|_L = \mathrm{id}$.

Consider the following symplectomorphisms of an odd symplectic supermanifold $E = \Pi T^*M$:

(3.12) symplectomorphisms induced by diffeomorphisms of M

(these symplectomorphisms preserve the cotangent bundle structure), symplectomorphisms "adjusted to M", i.e. identical on M:

$$\varphi: \quad \varphi^*\omega = \omega\,, \ \varphi\big|_M = \mathrm{id} \tag{3.13}$$

(they destroy the cotangent bundle structure except for $\varphi = \mathrm{id}$), symplectomorphisms corresponding to closed one-forms on M:

$$\varphi^* x^i = x^i\,, \ \varphi^*\theta_i = \theta_i + \alpha_i(x)\,, \ (\partial_i \alpha_j - \partial_j \alpha_i = 0) \tag{3.14}$$

where (x^i, θ_i) are coordinates on ΠT^*M corresponding to some coordinates (x^i) on M and $\alpha = \alpha_i(x)dx^i$ is a closed one-form on M with odd values (these symplectomorphisms move the Lagrangian surface M; notice that an arbitrary $(n.0)$-dimensional Lagrangian surface L is given by the equations $\theta_i - \alpha_i(x) = 0$ where $\alpha_i(x)dx^i$ is a closed odd-valued one-form). It might be worth noting that symplectomorphisms of ΠT^*M form a supergroup.

One can prove that an arbitrary symplectomorphism of ΠT^*M can be represented as the composition of symplectomorphisms (3.12), (3.13) and (3.14) (see [**9**]). (Compare this statement with the statement that an arbitrary transformation of Darboux coordinates can be represented as the composition of transformations (3.5), (3.6) and (3.7)).

Notice that every adjusted symplectomorphism (3.13) has the following appearance:

$$\begin{aligned} &x^i \to x^i + f^i(x,\theta)\,, \quad \text{where } f^i(x,\theta) = O(\theta) \\ &\theta_i \to \theta_i + g_i(x,\theta)\,, \quad \text{where } g_i(x,\theta) = O(\theta^2) \end{aligned} \tag{3.15}$$

where (x^i, θ_i) are the Darboux coordinates on ΠT^*M corresponding to coordinates x^i on M. One can show that there exists a Hamiltonian $Q(x,\theta) = Q^{ik}(x,\theta)\theta_i\theta_k$ that generates this transformation, i.e., (3.15) can be included in a 1-parameter family of transformations defined by the differential equation $\dot{z} = \{Q, z\}$ (see [**9**] for details).

Comparing (3.15) with (3.1) we arrive at an important conclusion:

PROPOSITION 1. *The top-degree component of the form corresponding to a semidensity on ΠT^*M does not change under any symplectomorphism adjusted to M. In other words, a semidensity on an odd symplectic supermanifold E defines a volume form (density) for all $(n.0)$-dimensional Lagrangian surfaces*[3].

Now consider a *closed* semidensity on ΠT^*M. To it corresponds a closed differential form on M. It follows from (3.9) and (3.13) that the action of an adjusted symplectomorphism (3.15) changes the semidensity and the corresponding form by an exact semidensity and an exact form respectively. Hence, we arrive at another important conclusion:

PROPOSITION 2. *To a closed semidensity on ΠT^*M corresponds a cohomology class of differential forms on M independently of the bundle structure. If two closed semidensities $\boldsymbol{s}$ and $\boldsymbol{s}'$ coincide on M and differ by an exact semidensity, then there exists an adjusted symplectomorphism $\varphi : \Pi T^*M \to \Pi T^*M$ such that $\varphi^*\boldsymbol{s} = \boldsymbol{s}'$.*

Propositions 1 and 2 were stated and proved in [**14**] and in [**9**], but in the work [**14**] semidensities do not appear explicitly.

Based on the concept of semidensities, the properties of the canonical Laplacian and the above Propositions we will now analyze the statements (2.14) concerning the Batalin–Vilkovisky formalism.

3.3. Master equation on semidensities. The claim is that *the Batalin–Vilkovisky master equation* (2.13) *is an equation on the semidensity $\boldsymbol{s} = \sqrt{f(x,\theta)\boldsymbol{\rho}}$. A solution of the Batalin–Vilkovisky quantum master equation is a closed semidensity:* $\Delta\boldsymbol{s} = 0$.

Suppose E is an odd symplectic supermanifold with the compact connected orientable underlying manifold M. E can be identified with ΠT^*L for every closed $(n.0)$-dimensional Lagrangian submanifold L (see [**9**] for details) and any two identifications differ by an adjusted symplectomorphism. An arbitrary $(n.0)$-dimensional closed Lagrangian surface L is given by a closed one-form on M (see (3.14)).

Let us now rewrite the implications (2.14) in terms of semidensities:

$$\boldsymbol{\rho} \text{ is a normal volume form} \;\Rightarrow\; \Delta\sqrt{\boldsymbol{\rho}} = 0 \;\Rightarrow\; \Delta_{\boldsymbol{\rho}}^2 = 0\,. \tag{3.16}$$

The first implication follows from the definition of the canonical operator Δ. The second implication follows from equation (3.10). Let us analyze to what extent these conditions are equivalent. Let $\boldsymbol{\rho}$ be a volume form such that $\Delta_\rho^2 = 0$. By (3.10), then it follows that $\Delta\sqrt{\boldsymbol{\rho}} = \nu\sqrt{\rho}$, where ν is an odd constant. (If external odd parameters are not allowed, then, of course, $\nu = 0$. Our analysis takes into consideration possible "odd moduli".) This odd constant is the obstruction to the condition $\Delta\sqrt{\boldsymbol{\rho}} = 0$, i.e. to the closedness of the semidensity $\sqrt{\boldsymbol{\rho}}$.

Suppose $\nu = 0$. Then the master equation $\Delta\sqrt{\boldsymbol{\rho}} = 0$ is satisfied. Consider in this case an arbitrary closed $(n.0)$-dimensional Lagrangian surface L and an arbitrary identifying symplectomorphism $\varphi\colon E \to \Pi T^*L$. Under the map (3.1) to every closed semidensity $\sqrt{\rho}$ on E corresponds a closed differential form $\omega = \omega_0 + \omega_1 + \cdots + \omega_n$ on L, where the top degree form ω_n defines a volume form on L. The closed 0-form ω_0 is a constant. It is easy to see from (3.1) and (3.14)

[3] A relation between semidensities on E and densities on Lagrangian surfaces can be defined for arbitrary Lagrangian surfaces (see [**14**] and [**7**]).

that the value of this constant (up to a sign) does not depend on the choice of the Lagrangian surface and on the choice of the identifying symplectomorphism. The top degree form, clearly, depends on the Lagrangian surface but does not depend on the identifying symplectomorphism (by Proposition 1). On the other hand, all other closed forms ω_k $(1 < k < n)$ can be eliminated by a suitable choice of the identifying symplectomorphism.

We have arrived, finally, to the following theorem:

THEOREM 1. *Let $E^{n.n}$ be an odd symplectic supermanifold with the closed orientable compact underlying manifold M. Let ρ be a volume form on $E^{n.n}$ such that $\Delta_\rho^2 = 0$. To this volume form corresponds an odd constant ν. If this odd constant is equal to zero, then the volume form defines a closed semidensity $\boldsymbol{s} = \sqrt{\rho}$, a solution of the Batalin–Vilkovisky quantum master equation. To a closed semidensity corresponds a constant c defined by the zero cohomology class of the differential form corresponding to the semidensity $\boldsymbol{s}$. If this constant is equal to zero, then the volume form $\boldsymbol{\rho}$ is normal.*

4. Discussion

The existence of Darboux coordinates that can locally make flat every surface in an odd symplectic supermanifold together with the absence of an invariant volume form make odd symplectic geometry a poor candidate for finding local invariants if no extra structure is provided. Hence the existence of the canonical odd Laplacian (3.2) looks mysterious. We shall to explain this fact briefly . (See details in [**10**])

Consider an arbitrary n-th order linear operator acting on functions or densities of some weight t on an arbitrary manifold (or supermanifold). One can consider its principal symbol, i.e., the coefficients at the highest order derivatives. It is a contravariant tensor field of rank n. In the case of an n-th order operator $\hat{A}$ acting on densities of weight t, the adjoint operator $\hat{A}^\dagger$ acting on densities of weight $1-t$ can be defined by the equation $\int \boldsymbol{s}_1 \cdot (\hat{A}\boldsymbol{s}_2) = \int (\hat{A}^\dagger \boldsymbol{s}_1) \cdot \boldsymbol{s}_2$, where $\boldsymbol{s}_2$ is an arbitrary density of weight t and $\boldsymbol{s}_1$ is an arbitrary density of weight $1-t$. Hence in the case of an operator $\hat{A}$ acting on semidensities $(t = 1/2)$ the operators $\hat{A}$ and $\hat{A}^\dagger$ act on the same space. Assuming that the coefficients are real, the operators $\hat{A}$ and $\hat{A}^\dagger$ have the same principal symbol. One can consider the principal symbol of $\hat{A} - \hat{A}^\dagger(-1)^n$, which is a tensor field of rank $n-1$. It is the so-called *subprincipal symbol* of the operator $\hat{A}$.

Let us consider the canonical operator Δ on an odd symplectic supermanifold in arbitrary coordinates (not necessary Darboux coordinates). The highest order coefficients make the principal symbol, which is here the tensor of rank 2 defining the odd symplectic structure. (More precisely it is the tensor $\mathfrak{S}^{AB}$ that defines the master Hamiltonian (1.8).) It is easy to see that if $\hat{A}$ is an arbitrary linear differential operator of the second order on semidensities having the principal symbol defined by the odd symplectic structure, then the subprincipal symbol of this operator is equal to $[\hat{A}, f] - \mathcal{L}_{\mathbf{D}_f}$. Hence it follows from (3.9) that the subprincipal symbol of the canonical Laplacian Δ is equal to zero. The coefficients at the first derivatives are fixed by this condition. In fact, these two conditions on the principal symbol and subprincipal symbol are equivalent to the equation (3.9). An arbitrary linear operator Δ' on semidensities obeying the condition (3.9) is equal

to $\Delta + C$, where C is a scalar (a zero-order operator). The condition $\Delta'\sqrt{\rho} = 0$ for an arbitrary normal volume form fixes this scalar $C = 0$ according to (3.11): hence, *on an odd symplectic supermanifold there is no distinguished volume form, but there is a distinguished class of normal volume forms.*

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Department of Mathematics, University of Manchester Institute of Science and Technology (UMIST), Manchester M60 1QD, UK

Laboratory of Computing Technique and Automation, Joint Institute for Nuclear Research, Dubna 141980, Russia

E-mail address: `khudian@umist.ac.uk`

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Differential operators and actions of Lie algebroids

Y. Kosmann-Schwarzbach and K. C. H. Mackenzie

ABSTRACT. We demonstrate that the notions of derivative representation of a Lie algebra on a vector bundle, of semi-linear representation of a Lie group on a vector bundle, and related concepts, may be understood in terms of representations of Lie algebroids and Lie groupoids, and we indicate how these notions extend to derivative representations of Lie algebroids and semi-linear representations of Lie groupoids in general.

Introduction

This paper deals with actions on vector bundles. The first part (Sections 1–3) deals with the infinitesimal actions of Lie algebras and Lie algebroids, while the second part (Sections 4 and 5) deals with the global actions of Lie groups and Lie groupoids.

When passing from the case of actions on vector spaces to that of actions on vector bundles, the notions of linear endomorphism and linear isomorphism admit straightforward generalizations which, however, have rarely been spelled out in the literature. On the global level, it is clear that the analogue of a linear isomorphism is a vector bundle automorphism, not necessarily base-preserving. Such an automorphism gives rise to an isomorphism of the vector space of sections which has the additional property of *semi-linearity*, *i.e.*, when a section is multiplied by a function on the base, the image of this section is multiplied by the given function composed with the inverse of the diffeomorphism of the base defined by the vector bundle automorphism. (This could be viewed as a morphism from the space of sections to itself, equipped with two module structures.) Actions of Lie groups on a vector bundle therefore give rise to representations in its space of sections by semi-linear isomorphisms.

The infinitesimal counterpart of such a vector bundle automorphism can be determined by differentiating a one-parameter group of semi-linear isomorphisms of the space of sections. The result is a *derivative endomorphism*, a first-order differential operator with the additional property that its symbol is a scalar multiple of the identity. Thus the derivative endomorphisms of the space of sections of a

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vector bundle generalize the linear endomorphisms of a vector space, and the representations of Lie algebras on the space of sections of a vector bundle by derivative endomorphisms generalize the linear representations.

These basic facts (and their cohomological interpretation, which we do not repeat here) were described by Kosmann-Schwarzbach in 1976 [**22**]. Actually, semi-linear isomorphisms and derivative endomorphisms are both particular cases of what Jacobson, in 1935, called *pseudo-linear transformations* [**19**], [**20**]. But Jacobson's purpose was to extend the theory of elementary divisors, and the relevance of his work for differential geometry was, to the best of our knowledge, forgotten.

Meanwhile, the theory of Lie groupoids was created by Ehresmann during the 1950s, and their infinitesimal counterpart, the Lie algebroids, were introduced by Pradines in 1967 [**35**]. Groupoid theory has been developed by many authors, in the context of homotopy theory and C^*–algebras, as well as in differential geometry, and it has come to differ in substantial respects from the elementary theory of groups. However, there is a Lie functor which associates to a Lie groupoid, G, a Lie algebroid, denoted AG, by a construction which extends in a straightforward way the construction of the Lie algebra of a Lie group. The resulting Lie theory provides a very broad framework which encompasses many of the infinitesimal constructions of differential geometry.

The concept of action of a Lie groupoid on a fibered manifold goes back to Ehresmann [**8**], who showed that when a groupoid acts on a space that is fibered over the base of the groupoid, the pullback of the groupoid by the fibration map is, in a natural way, a groupoid, called the *action groupoid* or *transformation groupoid.* The development of an abstract theory of Lie algebroids is more recent: in [**17**] Higgins and Mackenzie provided a basic set of algebraic constructions for abstract Lie algebroids modelled on those known in groupoid theory. In particular, they defined the infinitesimal action of a Lie algebroid on a fibered manifold and the associated *action Lie algebroid* or *transformation Lie algebroid.* The Lie algebroid of an action Lie groupoid is the action Lie algebroid of the corresponding infinitesimal action; however the action Lie algebroid of an infinitesimal Lie algebra action may integrate to a Lie groupoid which is not necessarily an action groupoid. This may at first seem to be a negative feature, but in fact the groupoid in such a situation provides a global object encoding the infinitesimal action even though a global action is not available. Here, as is generally the case with abstract Lie algebroids, determining the conditions under which a particular construction for Lie algebroids will admit a global analogue is a difficult problem. See Dazord [**7**], and Moerdijk and Mrčun [**33**] for a systematic account of the integration of Lie algebroid actions which includes many applications.

The aim of this paper is to relate the concept of Lie algebroid action to the earlier work [**22**] on representations by derivative endomorphisms (and, correspondingly, Lie groupoid actions and representations by semi-linear isomorphisms.) There are several aspects to the relationship. First, the derivative endomorphisms of the space of sections of a vector bundle are the sections of a Lie algebroid (Theorem 1.4). In [**30**], Mackenzie called this Lie algebroid the *Lie algebroid of covariant differential operators*, and denoted it by $\mathrm{CDO}(E)$, for a vector bundle E. Here, we denote it by $\mathfrak{D}(E)$. A *derivative representation* of a Lie algebra on a vector bundle is defined as a morphism into the Lie algebra of sections of $\mathfrak{D}(E)$, while a representation of a Lie algebroid on a vector bundle is a morphism into this Lie

algebroid. We prove that the derivative representations of Lie algebras on vector bundles coincide with the representations of action Lie algebroids (Theorem 2.4). Then, with a suitable definition of the derivative representations of a Lie algebroid on a vector bundle, we show that this one-to-one correspondence can be extended to the case of Lie algebroid actions (Theorem 3.5).

In Sections 4 and 5, we present global versions of the preceding results. We define the *semi-linear isomorphisms* of the space of sections of a vector bundle. A *semi-linear representation* of a Lie group on a vector bundle is defined as a morphism into the group of semi-linear isomorphisms of its space of sections, satisfying a smoothness condition, while a representation of a Lie groupoid on a vector bundle is a morphism of Lie groupoids into the Lie groupoid of all linear isomorphisms from a fiber to some (in general different) fiber of the bundle. In Theorem 4.8 we prove that the semi-linear representations of Lie groups on vector bundles coincide with the representations of their action groupoids. For a groupoid acting on a fibered manifold, F, there is likewise an action groupoid; considering an action of the groupoid on a vector bundle with base F, we show in Proposition 5.5 that there is an associated semi-linear representation which is a group representation of the group of bisections of the groupoid.

In the Appendix, we recall the definition of the twisted derivations of an algebra, and we show that the pseudo-linear endomorphisms of the module of sections ΓE of a vector bundle E with base M can be defined in terms of twisted derivations of an algebra whose underlying vector space is $C^\infty(M)\oplus\Gamma E$. In particular, the derivative endomorphisms can be defined in terms of the derivations of this algebra. Similarly, the semi-linear isomorphisms can be defined in terms of algebra automorphisms.

Throughout the paper, the manifolds that we consider are assumed to be smooth and second countable, and all maps are assumed to be smooth.

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1. Infinitesimal automorphisms of vector bundles

Consider a vector bundle (E, q, M) and let $C^\infty(M)$ be the algebra of $\mathbb{R}$–valued functions on M. We are concerned with first-order differential operators, $D\colon \Gamma E \to \Gamma E$, for which there exists a vector field D_M on M such that

$$D(f\psi) = fD(\psi) + D_M(f)\psi \ , \tag{1}$$

for all $f \in C^\infty(M)$ and all $\psi \in \Gamma E$. Below we explain that operators on modules over a ring satisfying (1), or some variant of it, have been studied under a variety of names. In the case of the module of sections of a vector bundle, we shall call such a differential operator a *derivative endomorphism* of ΓE.

DEFINITION 1.1. Let (E, q, M) be a vector bundle. A *derivative endomorphism* of ΓE is an $\mathbb{R}$–linear endomorphism D of ΓE such that there exists an $\mathbb{R}$–linear endomorphism, D_M, of $C^\infty(M)$ satisfying (1).

It follows that D is a first-order differential operator. It also follows that D_M is a derivation of $C^\infty(M)$, and therefore corresponds to a vector field on the base manifold M. The differential operator D is of order 0 if and only if $D_M = 0$.

EXAMPLE 1.2. Given any linear connection ∇ in a vector bundle E, and any vector field X on the base M, the covariant derivation ∇_X satisfies (1).

Recall that the *symbol*, $\sigma(D)$, of a first-order differential operator D is defined by

$$\sigma(D)(df) = [D, f] \ ,$$

where f is any smooth function on M and $[D, f]$ is the commutator of the operators D and multiplication by f. Condition (1) is equivalent to the requirement that the symbol of D, evaluated at any 1–form ξ on M at $x \in M$, be a scalar multiple of the identity of the fiber E_x at x, $\sigma(D)\xi = \langle \xi, D_M \rangle \mathrm{Id}_{E_x}$. Therefore, a first-order differential operator is a derivative endomorphism of ΓE if and only if it has scalar symbol.

Whereas the first-order differential operators on a vector bundle of rank > 1 do not constitute a Lie algebra under the commutator, a simple computation shows that the commutator

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1 \tag{2}$$

of two derivative endomorphisms is also a derivative endomorphism, and that

$$[D_1, D_2]_M = [(D_1)_M, (D_2)_M]. \tag{3}$$

It is also clear that the derivative endomorphisms form a module over $C^\infty(M)$. The $\mathbb{R}$–Lie algebra structure and the $C^\infty(M)$–module structure on the vector space of derivative endomorphisms of E interact according to a Leibniz rule,

$$[D_1, fD_2] = f[D_1, D_2] + (D_1)_M(f)D_2 \ , \tag{4}$$

which is a property characteristic of the sections of a Lie algebroid. We recall the concept of Lie algebroid, introduced by Pradines in [**35**].

DEFINITION 1.3. Let M be a manifold. A *Lie algebroid* on M is a vector bundle (A, q, M) together with a vector bundle map $a\colon A \to TM$ over M, called the *anchor* of A, and a bracket $[\ ,\ ]\colon \Gamma A \times \Gamma A \to \Gamma A$ which is $\mathbb{R}$–bilinear and alternating, satisfies the Jacobi identity, and is such that

$$a([X, Y]) = [a(X), a(Y)], \tag{5}$$

$$[X, fY] = f[X, Y] + a(X)(f)Y, \tag{6}$$

for all $X, Y \in \Gamma A$, $f \in C^\infty(M)$.

Given Lie algebroids A and A' on the same base M, a *Lie algebroid morphism from A to A'* is a vector bundle morphism $\varphi\colon A \to A'$ over M such that

$$a' \circ \varphi = a, \tag{7}$$

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)], \tag{8}$$

for all $X, Y \in \Gamma A$.

For Lie algebroids A and A' on different base manifolds, the notion of morphism is considerably more complicated; see [**17**].

Any Lie algebra is a Lie algebroid over $M = \{\cdot\}$, and for any manifold M, the tangent bundle TM is a Lie algebroid with anchor $a = \mathrm{Id}_{TM}$. Further examples will be introduced below.

THEOREM 1.4. *Given a vector bundle E on base M, there is a Lie algebroid $\mathfrak{D}(E)$ on M, the smooth sections of which are the derivative endomorphisms of ΓE, for which the bracket is the commutator bracket* (2) *and the anchor is the map $D \mapsto D_M$.*

PROOF. The first-order differential operators are the sections of a vector bundle $\mathrm{Diff}^1(E)$ on M. The symbol map is a vector bundle morphism, $\sigma\colon \mathrm{Diff}^1(E) \to \mathrm{Hom}(T^*M, \mathrm{End}(E))$, where $\mathrm{End}(E)$ is the vector bundle over M the fibers of which are the endomorphisms of the fibers of E. It is a surjective submersion and has as its kernel the differential operators of order 0. There is thus a short exact sequence of vector bundles over M,

$$\mathrm{End}(E) \overset{\subseteq}{\rightarrowtail} \mathrm{Diff}^1(E) \overset{\sigma}{\twoheadrightarrow} \mathrm{Hom}(T^*(M), \mathrm{End}(E)) \ .$$

Identifying $\mathrm{Hom}(T^*(M), \mathrm{End}(E))$ with $TM \otimes \mathrm{End}(E)$ in the canonical way, we define $\mathfrak{D}(E)$ to be the pullback vector bundle defined by the symbol map and the injection $TM \to TM \otimes \mathrm{End}(E)$, $X \mapsto X \otimes \mathrm{Id}_E$, according to the diagram:

$$\begin{array}{ccc} \mathfrak{D}(E) & \longrightarrow & TM \\ \downarrow & & \downarrow \\ \mathrm{Diff}^1(E) & \underset{\sigma}{\longrightarrow} & TM \otimes \mathrm{End}(E). \end{array}$$

The pullback exists because σ is a surjective submersion. Since the right-hand vertical arrow is an injective immersion, it follows that so is the left-hand arrow, and we can therefore regard $\mathfrak{D}(E)$ as a vector subbundle of $\mathrm{Diff}^1(E)$. Similarly, because σ is a surjective submersion, it follows that so is the top arrow, which we denote by a. Clearly the kernel of a is still End(E), and there is an exact sequence of vector bundles over M,

$$\mathrm{End}(E) \rightarrowtail \mathfrak{D}(E) \overset{a}{\twoheadrightarrow} TM,$$

where the sections of $\mathfrak{D}(E)$ are those first-order differential operators $D\colon \Gamma E \to \Gamma E$ for which there exists a vector field $D_M = a(D)$ on M such that (1) is satisfied.

We have already observed that for any pair of derivative endomorphisms D_1, D_2 in ΓE, the bracket (2) is also a derivative endomorphism of ΓE, with $a([D_1, D_2]) = [a(D_1), a(D_2)]$, and that (4) is satisfied. Thus the proof that $\mathfrak{D}(E)$ is a Lie algebroid on M with anchor a is complete. ∎

In general, a Lie algebroid is called *transitive* if the anchor is surjective. It was shown in the course of the proof of Theorem 1.4 that $\mathfrak{D}(E)$ is transitive. A right-inverse $\nabla\colon TM \to \mathfrak{D}(E)$ to the anchor is a linear connection in E in the standard sense. The curvature $R_\nabla\colon TM \oplus TM \to \mathrm{End}(E)$ of ∇ is given by

$$R_\nabla(X_1, X_2) = \nabla_{[X_1, X_2]} - [\nabla_{X_1}, \nabla_{X_2}]. \tag{9}$$

In particular, the connection is flat if and only if ∇ is a morphism of Lie algebroids.

Now consider the case of a trivial vector bundle, $E = M \times V$, and define a morphism from the vector bundle $TM \oplus (M \times \mathfrak{gl}(V))$ into $\mathfrak{D}(E)$ by

$$(X + u)(\psi) = X(\psi) + u(\psi)\colon M \to V \ ,$$

where $X(\psi)$ is the Lie derivative. For an arbitrary vector bundle, such a direct sum decomposition of $\mathfrak{D}(E)$ is given locally by a trivialization of E, while a global decomposition is defined by the choice of a linear connection, ∇, on E. Let ∇ also denote the extension of the chosen connection to the sections of $\mathrm{End}(E)$. Then for $D_1 = X_1 + u_1$ and $D_2 = X_2 + u_2$,

$$[D_1, D_2] = [X_1, X_2] + (\nabla_{X_1} u_2 - \nabla_{X_2} u_1 + [u_1, u_2] - R_\nabla(X_1, X_2)) \ .$$

By definition, when $R_\nabla = 0$, this bracket gives $TM \oplus \mathrm{End}(E)$ the *trivial Lie algebroid structure*. Thus the Lie algebroid $\mathfrak{D}(E)$ is trivial if and only if E admits a flat connection.

The following result follows from a theorem of Bourbaki [**4**, III.10.9, Prop. 14].

PROPOSITION 1.5. *The derivative endomorphisms of ΓE are the $\mathbb{R}$–linear endomorphisms of ΓE which can be extended to derivations of the algebra of sections of the tensor algebra of E.*

The notion of derivative endomorphism is in essence purely algebraic. Given a left module $\mathcal{E}$ over a ring $\mathcal{C}$, a *derivation of $\mathcal{E}$ over a derivation $D_\mathcal{C}$ of $\mathcal{C}$* is an additive map $D\colon \mathcal{E} \to \mathcal{E}$ such that

$$D(f\psi) = fD(\psi) + D_\mathcal{C}(f)\psi \tag{10}$$

for all $f \in \mathcal{C}$ and $\psi \in \mathcal{E}$. We can also refer to a derivation of $\mathcal{E}$ over a derivation $D_\mathcal{C}$ of $\mathcal{C}$ as a *derivative endomorphism* of $\mathcal{E}$. Note, however, that in the general case, D does not determine $D_\mathcal{C}$.

Bourbaki [**4**] defines a very general notion of derivation (involving six possibly different modules over a ring, three bilinear maps and three linear maps), and shows that the set of those derivations that belong to the "cas I" is closed under the commutator. The definition of the derivations in "cas I" includes the derivative endomorphisms as a particular case. The derivative endomorphisms of $\mathcal{E}$ are also a special case of the *pseudo-linear endomorphisms*, introduced and studied in the case of a finitely generated module over a ring by Jacobson in [**19**]. In fact, Jacobson uses the term 'vector space over a field' for module over a ring and the term *transformation* for endomorphism. His definition of the *pseudo-linear transformations* involves an automorphism $f \mapsto \overline{f}$ of $\mathcal{C}$ such that the following generalization of (10) is satisfied,

$$D(f\psi) = \overline{f}D(\psi) + D_\mathcal{C}(f)\psi.$$

This generalization is applicable, for instance, when one considers modules over rings of complex-valued functions, in which case $f \mapsto \overline{f}$ is the conjugation of complex numbers. Jacobson gave what amounts to the covariant derivation of vector fields in $\mathbb{R}^n$ as an example of such transformations, and derived the transformation law for connections under a change of basis. Then he studied the generalization of the theory of elementary divisors to the case of these pseudo-linear transformations. In [**20**], he called *differential transformations* those pseudo-linear transformations for which $f = \overline{f}$.

Working in this algebraic setting, Herz [**15**], [**16**], referring to Jacobson, showed that the differential transformations of a vector space over a commutative field form a *Lie pseudo-algebra*, a notion which he introduced. This may now be seen as an algebraic form of the notion of Lie algebroid in which vector bundles over manifolds are replaced by modules over rings, vector fields by derivations of rings, and so on.

Herz studied the sub-Lie-pseudo-algebras of a Lie pseudo-algebra over a skew field, and then showed that what he called the kernel of the Lie pseudo-algebra is a Lie algebra. (There is a corresponding result in the theory of Lie algebroids, that the kernel of the anchor in a transitive Lie algebroid is a Lie algebra bundle. However, the algebraic results proved for Lie pseudo-algebras do not automatically imply the corresponding theorems in the category of Lie algebroids, since submodules of finitely generated projective modules do not necessarily correspond to vector subbundles.)

A Lie pseudo-algebra that generalizes the Lie algebroid $\mathfrak{D}(E)$ to the more general setting of modules over Lie pseudo-algebras was constructed in [**29**]; the construction is also given in [**18**]. For references dealing with Lie pseudo-algebras, also now known as *Lie-Rinehart algebras*, see [**31**].

While the algebraic properties of covariant derivatives were already stated explicitly by Schouten in 1924, in his book, *Ricci Kalkül*, the algebraization of the theory of connections is due to Koszul [**25**], and a very general formalization was later accomplished by a student of Ehresmann, C. M. de Barros, in his thesis [**2**]. There, he introduced the *translations infinitésimales graduées*, of which the derivative endomorphisms are a particular case, and he studied the *représentations infinitésimales graduées*, which generalize the *lois de dérivation* of Koszul. (We shall deal with such Lie algebra representations in the next section.) When the module being considered is that of the sections of a vector bundle, the derivative endomorphisms and their higher-order analogues were introduced by Palais [**34**] under the name *quasi-scalar differential operators* in his analysis of elliptic self-adjoint operators on Riemannian vector bundles. At about the same time, Ngô Van Que [**36**], [**37**], [**38**] used operators satisfying (1) in his work on Lie groupoids. In [**22**], the name *derivative endomorphisms* of ΓE for operators satisfying (1) was introduced, and it was remarked that such operators are the first-order differential operators on E with scalar symbol. Such operators were independently introduced in [**30**], [**31**] where, because of the important example 1.2, they are called *covariant differential operators*. A number of papers [**3**], [**39**] introduce an *Atiyah algebra* of a vector bundle E, which is again the module of sections of $\mathfrak{D}(E)$.

Derivative endomorphisms arise whenever the infinitesimal of a representation of a group defined by an action on a vector bundle is considered. In [**14**], Hermann introduced the first-order differential operators satisfying relation (1) and he briefly indicated "the geometric genesis of these operators". In [**22**], the *semi-linear endomorphisms* of ΓE were introduced (see Section 4) and it was shown that an action of a Lie group on the sections of a vector bundle by semi-linear transformations differentiates to an action of its Lie algebra by differential operators which are derivative endomorphisms. It was stated (p. 86) that the Lie algebra of derivative endomorphisms of ΓE is to be considered "as the Lie algebra of the 'infinite-dimensional Lie group' of all automorphisms of the vector bundle E". One may say that a derivative endomorphism of ΓE is an *infinitesimal automorphism* of the vector bundle E, in the same way that a vector field is an infinitesimal automorphism of a manifold. Derivative endomorphisms play a crucial role throughout the theory of Lie algebroids; it was argued in detail in [**30**] that $\mathfrak{D}(E)$ is the correct generalization to vector bundles of the general linear Lie algebra $\mathfrak{gl}(V)$ of a vector space V. Theorem 1.4 was proved in [**30**]. We will discuss the relationships between these approaches further below.

2. Derivative representations of Lie algebras and action Lie algebroids

We again consider a vector bundle (E, q, M). Since the derivative endomorphisms of ΓE play the role of the infinitesimal automorphisms of the vector bundle E, it is natural to define:

DEFINITION 2.1. A *derivative representation* of a Lie algebra $\mathfrak{g}$ on E is a morphism of $\mathfrak{g}$ into the Lie algebra $\Gamma\mathfrak{D}(E)$.

Associated to a derivative representation, ρ, of $\mathfrak{g}$ on E, there is an infinitesimal action of $\mathfrak{g}$ on E, $X \mapsto (\rho(X))_M$. Of course, if the base M of E is a point, we recover the usual notion of linear representation.

EXAMPLE 2.2. Let us consider invariant prequantization as defined in [**13**]. Let (M, ω) be a symplectic manifold, and suppose that ω is integral. Then there is a complex line bundle E on M, together with a Hermitian fiber-metric, and a connection ∇ in E with respect to which the metric is parallel, and such that the curvature R_∇ of ∇ satisfies

$$R_\nabla = -i\omega.$$

The prequantization action of $C^\infty(M)$ on sections of E is the map, $\delta\colon C^\infty(M) \to \Gamma\mathfrak{D}(E)$, defined by

$$\delta(f)(\psi) = \nabla_{X_f}(\psi) + if\psi,$$

where $f \in C^\infty(M)$, $\psi \in \Gamma E$, and where X_f is the Hamiltonian vector field of f. It maps the Poisson bracket to the Lie algebroid bracket of $\mathfrak{D}(E)$, respects the anchors in the sense that $a(\delta(f)) = X_f$ for $f \in C^\infty(M)$, and takes values in the Lie subalgebroid $\mathfrak{D}_0(E)$ of $\mathfrak{D}(E)$, whose sections respect the metric.

Now suppose that $G \times M \to M$ is a Hamiltonian action of a Lie group G on M, with a bracket-preserving moment map, $\widehat{J}\colon \mathfrak{g} \to C^\infty(M)$. Then $X_{\widehat{J}(X)} = X_M$, for all $X \in \mathfrak{g}$, where $X \mapsto X_M$, $\mathfrak{g} \to \Gamma TM$, is the infinitesimal action. Now $\rho = \delta \circ \widehat{J}\colon \mathfrak{g} \to \Gamma\mathfrak{D}(E)$ is a derivative representation of $\mathfrak{g}$ on E.

There is also a well-known notion of a Lie algebroid representation, which we recall.

DEFINITION 2.3. A *representation* of a Lie algebroid A, also on base M, on the vector bundle E is a Lie algebroid morphism from A to $\mathfrak{D}(E)$.

If the base M is a point, A is merely a Lie algebra and E is a vector space, and this definition reduces to that of a linear representation.

To express the relationship between these notions, we recall the definition of an *action Lie algebroid.* Consider an infinitesimal action of a Lie algebra $\mathfrak{g}$ on a manifold M, *i. e.*, a map $X \mapsto X_M$, $\mathfrak{g} \to \Gamma TM$, which is $\mathbb{R}$–linear, and preserves brackets, $[X, Y]_M = [X_M, Y_M]$ for all $X, Y \in \mathfrak{g}$. Extend this notation to maps $V\colon M \to \mathfrak{g}$ so that V_M is the vector field on M defined by $V_M(m) = (V(m)_M)(m)$, for $m \in M$. Then the trivial vector bundle $M \times \mathfrak{g}$ on M acquires a Lie algebroid structure with anchor $a\colon M \times \mathfrak{g} \to TM$ defined by $a(m, X) = X_M(m)$, and bracket

$$[V, W] = V_M(W) - W_M(V) + [V, W]^\bullet, \tag{11}$$

where $V_M(W)$ denotes the Lie derivative with respect to V_M of the vector-valued function W, and $[V, W]^\bullet$ is the pointwise bracket of maps into $\mathfrak{g}$. With this Lie algebroid structure, the trivial vector bundle $M \times \mathfrak{g}$ is called the *action Lie algebroid* corresponding to the infinitesimal action $X \mapsto X_M$, and we denote it by $\mathfrak{g} \lessdot M$.

Observe that if $V, W\colon M \to \mathfrak{g}$ are constant maps, then the Lie algebroid bracket $[V, W]$ is constant also, and is constant at the bracket in $\mathfrak{g}$ of the values of V and W. In fact this property, together with the anchor $a\colon \mathfrak{g} \lessdot M \to TM$ and (6), determines (11).

We can now relate the two concepts in Definitions 2.1 and 2.3.

THEOREM 2.4. *Let $X \mapsto X_M$ be an infinitesimal action of a Lie algebra $\mathfrak{g}$ on M. There is a one-to-one correspondence between derivative representations of $\mathfrak{g}$ on E, associated to the given infinitesimal action, and representations of the action Lie algebroid $\mathfrak{g} \lessdot M$ on E.*

PROOF. Let $\rho\colon \mathfrak{g} \to \Gamma\mathfrak{D}(E)$ be a derivative representation of $\mathfrak{g}$ associated to the given infinitesimal action $X \mapsto X_M$. By definition, the anchor of $\rho(X)$ is X_M. We define a vector bundle morphism σ from $M \times \mathfrak{g}$ to $\mathfrak{D}(E)$ by

$$\sigma(m, X) = (\rho(X))(m). \tag{12}$$

This vector bundle morphism is a morphism of Lie algebroids from $\mathfrak{g} \lessdot M$ to $\mathfrak{D}(E)$ because (i) it clearly satisfies the condition on the anchors, and (ii) the bracket condition is satisfied for constant maps from M to $\mathfrak{g}$. Conversely, if σ is a representation of $\mathfrak{g} \lessdot M$ on E, formula (12) defines a morphism ρ from $\mathfrak{g}$ to $\Gamma\mathfrak{D}(E)$. ∎

The significance of Theorem 2.4 is that, on the one hand, it can be immediately generalized once we extend the definition of action Lie algebroid to the case of Lie algebroids that are not Lie algebras, and on the other hand, when the given actions globalize, it admits a global formulation. We deal with these two aspects in the following sections.

EXAMPLE 2.5. If ρ is the map defined in Example 2.2, the map σ defined by (12) is a representation of $\mathfrak{g} \lessdot M$ on E, more precisely, a morphism of Lie algebroids from $\mathfrak{g} \lessdot M$ to the Lie subalgebroid $\mathfrak{D}_0(E)$ of $\mathfrak{D}(E)$.

EXAMPLE 2.6. Given any principal bundle $P(M, H)$ and a linear representation of H on a vector space V, let $E = P \times_H V$ be the associated vector bundle. Let $A = TP/H$ be the *Atiyah Lie algebroid* of the principal bundle. There is a representation σ of A on E defined as follows (see, *e.g.*, [**30**, App.A]). A section ψ of E can be identified with an H–equivariant map, $\widetilde{\psi}\colon P \to V$, and a section X of TP/H can be identified with an H–invariant vector field $\widetilde{X}$ on P. Since the Lie derivative of $\widetilde{\psi}$ with respect to $\widetilde{X}$ is also an H–equivariant map from P to V, it corresponds to a section $\sigma(X)(\psi)$.

If, in particular, P is a Lie group G with H a closed subgroup, this construction associates to any linear representation of H on V the representation σ of TG/H on $E = G \times_H V$. On the other hand, the Lie algebroid TG/H is canonically isomorphic to the action Lie algebroid $\mathfrak{g} \lessdot (G/H)$ arising from the standard action of $\mathfrak{g}$ on $M = G/H$, under the map $TG/H \to \mathfrak{g} \lessdot (G/H)$ defined by the canonical right identificatiion of TG with $G \times \mathfrak{g}$. It follows by Theorem 2.4 that there is a one-to-one correspondence between the linear representations of $\mathfrak{h}$ on V and the derivative representations of $\mathfrak{g}$ on $E = G \times_H V$.

In this case, the derivative representation of $\mathfrak{g}$ on $E = G \times_H V$ associated to the representation σ of $A = TG/H$ on E is the infinitesimal of the linear representation of G induced from that of H. So this result is the infinitesimal counterpart of an

observation that was made in [**22**] relating semi-linear representations of Lie groups to induced representations.

EXAMPLE 2.7. A representation of a Lie algebroid A on a vector bundle E extends to a representation by derivations of its exterior algebra, that is, a morphism from the $\mathbb{R}$–Lie algebra ΓA to the $\mathbb{R}$–Lie algebra of derivations of $\Gamma(\bigwedge E)$, and conversely the restriction to ΓE of such a representation by derivations of $\Gamma(\bigwedge E)$ is a representaion of A on E in the sense of Definition 2.3.

REMARK 2.8. Consider a Lie algebroid A and a vector bundle E on the same base M. Several authors [**40**] [**10**] consider the *A–connections on E*, defined as the vector bundle morphisms $A \to \mathfrak{D}(E)$ satisfying condition (7) on the anchors, without a bracket condition. If $A = TM$, this is precisely the standard concept of linear connection in E.

The *curvature* of such an A–connection ∇ is defined as in (9). Now a representation of A on E is precisely a flat A–connection on E. Extensive use of the notion of A–connection and of representation of Lie algebroids was made in [**40**] [**10**], especially the representations of A on $\bigwedge^n A$ where n is the rank of A. The one-to-one correspondence described in the proof of Theorem 2.4 is actually the restriction to the derivative endomorphisms of a one-to-one correspondence between derivative pre-representations of $\mathfrak{g}$ on E, where we only require linearity of the map $\mathfrak{g} \to \Gamma\mathfrak{D}(E)$, and A–connections on E.

The vector bundle that we have called the Atiyah Lie algebroid of a principal bundle was introduced by Atiyah in 1957 [**1**], ten years before Pradines introduced the abstract concept of Lie algebroid. It is also called the *gauge algebroid* of the principal bundle, and its sections are then called *infinitesimal gauge transformations* of the bundle. The definition of derivative representation comes from [**22**]. The derivative representations are a particular case of the graded infinitesimal representations of de Barros [**2**]. When $\mathfrak{g}$ is the Lie algebra of derivations of $C^\infty(M)$ and when the map $\mathfrak{g} \to \Gamma\mathfrak{D}(E)$ is $C^\infty(M)$–linear, a derivative representation is a flat derivation law in the sense of Koszul [**25**]. The definition of Lie algebroid representation may be found in [**30**, III.2.9]; it was probably first formulated in [**38**]. The corresponding algebraic notion is that of a left module over a Lie pseudo-algebra.

3. Lie algebroid actions and representations

In this section we first consider the infinitesimal actions of a Lie algebroid A with base M on a fibered manifold with base M, that is, on a surjective submersion onto M. Such an infinitesimal action of $A \to M$ on $\varphi\colon F \to M$ defines a Lie algebroid structure on the vector bundle φ^*A, the pullback of A, which is called the action Lie algebroid defined by the given infinitesimal action of A on F.

We then extend the definition of the derivative representations to the case of a Lie algebroid A and a vector bundle with base F. As a particular case of this notion, we recover Definition 2.1 in the case where M is a point. Finally in this section we show that the one-to-one correspondence between derivative representations and representations of action Lie algebroids can be extended to this more general situation.

DEFINITION 3.1. Let A be a Lie algebroid on M, and let $\varphi\colon F \to M$ be a fibered manifold. An *infinitesimal action of A on F* is an $\mathbb{R}$–linear map $X \mapsto X_F$, $\Gamma A \to \Gamma TF$, such that

(i) for each $X \in \Gamma A$, X_F is projectable to $a(X)$,
(ii) the map $X \mapsto X_F$ preserves brackets,
(iii) the map $X \mapsto X_F$ is $C^\infty(M)$–linear in the following sense: for each $f \in C^\infty(M)$ and each $X \in \Gamma A$,

$$(fX)_F = (f \circ \varphi)X_F. \tag{13}$$

If ΓTF is given the $C^\infty(M)$–module structure defined by $(f, Y) \mapsto (f \circ \varphi)Y$, $C^\infty(M) \times \Gamma TF \to \Gamma TF$, then Equation (13) is the condition that $X \mapsto X_F$ is a morphism of left modules.

If, in particular, the base M is a point, then A is a Lie algebra, and we recover the notion of infinitesimal action of a Lie algebra on a manifold F, conditions (i) and (iii) being trivially satisfied.

If $F = M$ and φ is the identity, an infinitesimal action of the Lie algebroid A on the manifold M is a $C^\infty(M)$–linear map $X \mapsto X_M$ from ΓA to ΓTM, *i.e.*, a morphism of vector bundles from A to TM, which by condition (i) is necessarily the anchor of A.

It is shown in [**17**] that whenever an infinitesimal action of A on $\varphi\colon F \to M$ is defined, there is an associated Lie algebroid structure on the pullback vector bundle, φ^*A,

$$\begin{array}{ccc} \varphi^*A & \longrightarrow & A \\ \downarrow & & \downarrow \\ F & \xrightarrow[\varphi]{} & M. \end{array}$$

With this Lie algebroid structure, φ^*A is called the *action Lie algebroid* associated to the infinitesimal action $X \mapsto X_F$, and it is denoted $A \prec F$. We recall the definition of this Lie algebroid structure. We identify the module of sections of the pullback with $C^\infty(F) \otimes \Gamma A$, where the tensor product is over $C^\infty(M)$. Thus $h \otimes fX = h(f \circ \varphi) \otimes X$ and $h \otimes X$ is identified with $h(X \circ \varphi)$, for $h \in C^\infty(F)$, $f \in C^\infty(M)$, $X \in \Gamma A$. Then φ^*A is a Lie algebroid with the following anchor, a_F,

$$a_F(h \otimes X) = hX_F\ , \tag{14}$$

and $\mathbb{R}$–bilinear bracket,

$$[h \otimes X, k \otimes Y] = hk \otimes [X, Y] + hX_F(k) \otimes Y - kY_F(h) \otimes X, \tag{15}$$

where $h, k \in C^\infty(F)$, $X, Y \in \Gamma A$.

If the base M is a point, this Lie algebroid over F is merely the product $F \times A$ with the action Lie algebroid structure over base F defined in Section 2.

The notion of action and the construction of an action Lie algebroid extend to the situation in which φ is an arbitrary smooth map, but we shall not need this generality here.

EXAMPLE 3.2. Let A be any Lie algebroid on a connected base M, and let $f\colon \widetilde{M} \to M$ be any covering, with group π. Then there is a canonical infinitesimal action of A on $\widetilde{M}$ in which $X_{\widetilde{M}}$ is the π–invariant lift of $a(X)$ to $\widetilde{M}$.

EXAMPLE 3.3. Given any principal bundle $P(M, H)$, the Atiyah Lie algebroid TP/H with base M acts on $P \to M$ by $X \mapsto \overrightarrow{X}$, where $\overrightarrow{X}$ is the right-invariant

vector field on P defined by $X \in \Gamma(TP/H)$; the action Lie algebroid is canonically isomorphic to TP.

In view of Theorem 2.4, we shall define the notion of derivative representation of a Lie algebroid A with base M on a vector bundle E with base F, associated to an infinitesimal action of A on F, in such a way that the one-to-one correspondence between derivative representations and representations of the action Lie algebroid is extended.

DEFINITION 3.4. Let A be a Lie algebroid on M, and let $\varphi\colon F \to M$ be a fibered manifold. Let $X \mapsto X_F$, $\Gamma A \to \Gamma TF$, be an infinitesimal action of A on F. Let us consider a vector bundle E on F. A *derivative representation of A on E* associated to the infinitesimal action $X \mapsto X_F$ is a morphism, ρ, of Lie algebras from ΓA to $\Gamma\mathfrak{D}(E)$ such that
(i) for any $X \in \Gamma A$,

$$(\rho(X))_F = X_F \ , \tag{16}$$

(ii) ρ is $C^\infty(M)$–linear in the following sense: for each $f \in C^\infty(M)$ and each $X \in \Gamma A$,

$$\rho(fX) = (f \circ \varphi)\rho(X) \ . \tag{17}$$

Since for any $X \in \Gamma A$, $\rho(X)$ is a section of $\mathfrak{D}(E)$, condition (16) expresses the fact that, for any $h \in C^\infty(F), \psi \in \Gamma E$,

$$\rho(X)(h\psi) = h\rho(X)(\psi) + X_F(h)\psi. \tag{18}$$

THEOREM 3.5. *Let A, F, $X \mapsto X_F$ and E be as in Definition* 3.4. *There is a one-to-one correspondence between derivative representations of A on E associated to the infinitesimal action $X \mapsto X_F$ and representations of the action Lie algebroid $A \ltimes F$ on E.*

PROOF. We sketch the proof which is very similar to that of Theorem 2.4. If $\rho : \Gamma A \to \Gamma\mathfrak{D}(E)$ is a derivative representation of A on E associated to the given infinitesimal action of A on F, we define $\sigma\colon \varphi^*A \to E$ in terms of sections of the form $h \otimes X$, $h \in C^\infty(F), X \in \Gamma A$, by

$$\sigma(h \otimes X)(p) = h(p)\rho(X)(p) \ ,$$

for any $p \in F$. Then σ is a representation of the Lie algebroid φ^*A on E.

Conversely, if σ is a representation of the Lie algebroid φ^*A on E, we set, for $X \in \Gamma A$, and $p \in F$,

$$\rho(X)(p) = \sigma(X \circ \varphi)(p) \ .$$

Then ρ is a derivative representation of A on E. ∎

Thus we have obtained a generalization of Theorem 2.4 to the case of Lie algebroids acting on vector bundles.

The Lie algebroid structure (15) of an action Lie algebroid corresponds in the algebraic setting treated by Fel'dman [**11**] to a *crossed product* structure on its space of sections.

4. Semi-linear representations of Lie groups and action groupoids

In this section we provide a global formulation in terms of Lie groupoids for the construction described in Section 2: the relationship between derivative representations of Lie algebras on vector bundles and action Lie algebroids has a global analogue, the relationship between the semi-linear representations of Lie groups acting on vector bundles and the action groupoids, described in Theorem 4.8 below. In the following section we shall discuss the more general case of a global formulation for Lie algebroid representations. These global constructions are such that applying the Lie functor to the Lie groups and Lie groupoids under consideration yields the known infinitesimal constructions for Lie algebras and Lie algebroids.

For the basic definitions of groupoid theory used in what follows, see [**30**] or [**28**] or [**6**] (note that the composition convention in [**6**] is the opposite to that followed here). Given a vector bundle (E, q, M), we denote by $\Phi(E)$ the set of all linear isomorphisms from a fiber of E to some (in general different) fiber of E. Then $\Phi(E)$ is a Lie groupoid with base M, with its groupoid structure arising from the ordinary composition of maps, and its smooth structure given by locally identifying $\Phi(M \times V)$ with $M \times GL(V) \times M$. (In [**30**], this groupoid is called the *linear frame groupoid* of E.) The theorem which follows is fundamental to the results of this section.

THEOREM 4.1. *The Lie algebroid of the Lie groupoid $\Phi(E)$ is canonically isomorphic to $\mathfrak{D}(E)$.*

Theorem 4.1 emerged from two separate lines of work. On the one hand, a proof was given by Kumpera [**26**] in unpublished notes of 1971, but was not included in the published version (Appendix A of [**27**]). A simplification of Kumpera's proof was given by Mackenzie in 1987 [**30**, III.4.5]. On the other hand, it was remarked by Hermann in [**14**] and independently by Kosmann-Schwarzbach in [**22**] that the derivative endomorphisms of the space of sections of a vector bundle are the infinitesimal generators of one-parameter groups of vector bundle automorphisms. This leads precisely to the result that $A\Phi(E) \cong \mathfrak{D}(E)$.

We now define the semi-linear isomorphisms of ΓE and correspondingly the semi-linear representations of a Lie group on the vector bundle, E.

DEFINITION 4.2. Let (E, q, M) be a vector bundle. A *semi-linear isomorphism* of ΓE is an $\mathbb{R}$–linear automorphism μ of ΓE such that there exists an $\mathbb{R}$–linear isomorphism, μ^M, of $C^\infty(M)$ satisfying

$$\mu(f\psi) = \mu^M(f)\ \mu(\psi)\ , \tag{19}$$

for all $f \in C^\infty(M)$ and all $\psi \in \Gamma E$.

It follows that μ^M is a ring automorphism of $C^\infty(M)$. Any automorphism ν of the vector bundle E that projects onto a transformation ν_M of the base manifold gives rise to a semi-linear isomorphism $\psi \mapsto \nu \cdot \psi$ of ΓE defined by

$$\nu \cdot \psi = \nu \circ \psi \circ (\nu_M)^{-1}\ , \tag{20}$$

in which case the ring endomorphism of $C^\infty(M)$ is $f \mapsto f \circ (\nu_M)^{-1}$, and conversely, every semi-linear isomorphism of ΓE arises from a vector bundle automorphism of E. (See [**21**] [**22**].)

PROPOSITION 4.3. *Formula* (20) *establishes a one-to-one correspondence between vector bundle automorphisms of* E *and semi-linear isomorphisms of* ΓE.

The semi-linear isomorphisms of ΓE are the $\mathbb{R}$–isomorphisms of ΓE which can be extended to automorphisms of the algebra of sections of the tensor algebra of E.

DEFINITION 4.4. A *semi-linear representation* of a Lie group G on E is a morphism, R, from G into the group of semi-linear isomorphisms of ΓE such that, for any $\psi \in \Gamma E$, the map $(g, m) \mapsto R(g)(\psi)(m)$, $G \times M \to E$, is smooth.

Each semi-linear representation, R, of G on E is associated to an action $g \mapsto g_M$ of G on M such that $(R(g))^M(f) = f \circ (g_M)^{-1}$.

There is also a well-known notion of a Lie groupoid representation, which we recall.

DEFINITION 4.5. A *representation* of a Lie groupoid G with base M on the vector bundle E is a base-preserving Lie groupoid morphism from G to $\Phi(E)$.

If the base M is a point, G is merely a Lie group and E is a vector space, and this definition reduces to that of a linear representation. In general it follows from Theorem 4.1 that applying the Lie functor to a representation of G on E yields a representation of AG on E.

To express the relationship between the notion of semi-linear representation and that of a groupoid representation, we recall the definition of *action groupoid*. Consider an action $g \mapsto g_M$ of a Lie group G on a manifold M. The product manifold $G \times M$ becomes a Lie groupoid with base M when it is equipped with the source map $(g, m) \mapsto m$, the target map $(g, m) \mapsto g_M(m)$, for $g \in G$, $m \in M$, the partial multiplication $((g_1, m_1), (g_2, m_2)) \mapsto (g_1 g_2, m_2)$ defined if and only if $m_1 = (g_2)_M(m_2)$, and the inversion $(g, m) \mapsto (g^{-1}, g_M(m))$. With this Lie groupoid structure, the product manifold $G \times M$ is called the *action groupoid* corresponding to the action $g \mapsto g_M$, and we denote it by $G \ltimes M$. The following result is proved in [**17**, Th. 2.5].

PROPOSITION 4.6. *If* $\mathfrak{g}$ *is the Lie algebra of the Lie group* G, *the Lie algebroid of the action groupoid* $G \ltimes M$ *is the action Lie algebroid* $\mathfrak{g} \ltimes M$ *associated to the infinitesimal of the action of* G.

EXAMPLE 4.7. The representation σ of the Lie algebroid $\mathfrak{g} \ltimes M$ on E of Example 2.5 can be integrated to a morphism $G \ltimes M \to \Phi_0(E)$, where $\Phi_0(E)$ is the Lie subgroupoid of $\Phi(E)$ of isometries between the fibers of E, *i.e.*, to an orthogonal representation of $G \ltimes M$ on E, if and only if (E, ∇) together with the metric constitute G–invariant data in the sense of [**13**].

We can now relate the two concepts defined in 4.4 and 4.5.

THEOREM 4.8. *Let* $g \mapsto g_M$ *be an action of a Lie group* G *on* M. *There is a one-to-one correspondence between semi-linear representations of* G *on* E, *associated to the given action, and representations of the action groupoid* $G \ltimes M$ *on* E.

PROOF. The proof follows from the observations that semi-linear representations are in one-to-one correspondence with the actions of G on E by vector bundle automorphisms, which in turn are in one-to-one correspondence with morphisms of Lie groupoids from the action groupoid, $G \ltimes M$, to $\Phi(E)$, *i.e.*, representations of the action groupoid on E [**30**]. ∎

In view of the properties of the Lie functor stated in Theorem 4.1 and Proposition 4.6, this theorem implies Theorem 2.4 when the action of $\mathfrak{g}$ on M can be integrated globally.

REMARK 4.9. The semi-linear transformations of ΓE are another particular case of the pseudo-linear transformations of Jacobson [**20**], in which the linear mapping $D_{\mathcal{C}}$ vanishes, and the ring automorphism $f \mapsto \bar{f}$ of $\mathcal{C} = C^\infty(M)$ is, in the notation of Definition 4.2, the map μ^M.

REMARK 4.10. The isomorphism of Theorem 4.1 arises as an infinitesimal version of the correspondence between vector bundle automorphisms and semi-linear isomorphisms (see Proposition 4.3), but also admits another formulation.

A vector field X on a vector bundle (E, q, M) is *linear* if the flow of X consists of local vector bundle automorphisms of E. Such a vector field clearly projects to a vector field X_M on M. In [**32**], linear vector fields are characterized as pairs $X \in \Gamma TE$, $X_M \in \Gamma TM$ which constitute a vector bundle morphism from (E, q, M) to $(TE, T(q), TM)$.

The one-to-one correspondence between linear vector fields on E and derivative endomorphisms of ΓE can be expressed directly as follows [**23**] [**32**]. Given a vector field X on E which projects to a vector field X_M on M, and a section ψ of E, we define

$$D_X(\psi)(m) = T_m(\psi)(X_M(m)) - X(\psi(m)), \tag{21}$$

for $m \in M$. (Here the right-hand side is a vertical tangent vector, which we regard as an element of E in the usual way.) When X is linear, the $\mathbb{R}$–linear map D_X is a derivative endomorphism of ΓE associated with X_M, and is called the *Lie derivation* with respect to X. Conversely, given a derivative endomorphism D of ΓE associated with a derivation X_M of $C^\infty(M)$, we set

$$X_D(p) = T_m(\psi)(X_M(m)) - (D\psi)(m), \tag{22}$$

where $p \in E$ and ψ is a section of E such that $\psi(m) = p$. Then X_D is a well-defined linear vector field on E, and the maps $X \mapsto D_X$ and $D \mapsto X_D$ are obviously inverses of one another. Moreover this one-to-one correspondence preserves brackets. In fact, the linear vector fields are the section of a Lie algebroid $T^{\mathrm{LIN}}E$ with base M, defined in [**32**], and this correspondence arises from a Lie algebroid isomorphism from $T^{\mathrm{LIN}}E$ to $\mathfrak{D}(E)$. Therefore the isomorphism of Theorem 4.1 may also be formulated as an isomorphism between the Lie algebroid of $\Phi(E)$ and $T^{\mathrm{LIN}}E$.

An infinitesimal action of a Lie algebroid A on a vector bundle E in the sense of Definition 3.1 is called *linear* if the action is by linear vector fields. Since a Lie algebroid morphism from A to $T^{\mathrm{LIN}}E$ can be identified with a Lie algebroid morphism from A to $\mathfrak{D}(E)$, it follows that the linear infinitesimal actions of A on E can be identified with the representations of A on E in the sense of Definition 2.3. Since, by condition (i) of Definition 3.1, for each $X \in \Gamma A$, $X_M = a(X)$, the correspondence established by (21) coincides with that of [**17**, p. 212].

A more general non-linear version of the bijective correspondence between vector fields on E and differential operators on the sections of E was established in [**24**]. Consider a fiber bundle (F, q, M). Then, given an infinitesimal automorphism of the bundle, that is, a vector field X on F which projects to a vector field X_M on M, and a section ψ of F, formula (21) defines a vertical tangent vector to F, and we thus obtain a section of $\psi^* VF$, where VF is the vertical tangent bundle

of F. In this way one obtains an operator which is a *first-order differential section operator* from F to VF in the terminology of [**34**]. Conversely, such an operator corresponds to an infinitesimal automorphism if and only if it is quasi-scalar. For a fibered manifold (F, q, M), the set of infinitesimal automorphisms has both a module structure over $C^\infty(M)$ and a bracket, making it a Lie pseudo-algebra, but in general there is no underlying Lie algebroid.

5. Groupoid actions and representations

More generally, we now consider the action of a Lie groupoid on a fibered manifold. (We shall not consider the still more general case of the action of a Lie groupoid on an arbitrary smooth map. For this, see both [**17**] and the recent account of Moerdijk and Mrčun [**33**].)

Let G be a Lie groupoid with base M and source map α, and let $\varphi : F \to M$ be a fibered manifold. We denote by φ^*G the pullback in the following commutative diagram,

$$\begin{array}{ccc} \varphi^*G & \longrightarrow & G \\ \downarrow & & \downarrow \alpha \\ F & \xrightarrow[\varphi]{} & M. \end{array}$$

For $m \in M$, we denote $\varphi^{-1}(m) \subseteq F$ by F_m.

DEFINITION 5.1. Let G be a Lie groupoid with base M, source map α and target map β, and let $\varphi : F \to M$ be a fibered manifold. An *action* of G on F is a map, S, from G to the set of smooth bijective maps from a fiber of F to some (in general different) fiber of F, such that
(i) for any $g \in G$, $S(g)$ is a map from $F_{\alpha(g)}$ to $F_{\beta(g)}$,
(ii) for any $g_1 \in G$ and $g_2 \in G$ such that $\beta(g_1) = \alpha(g_2)$, $S(g_2 g_1) = S(g_2) \circ S(g_1)$,
(iii) the map $(g, p) \mapsto S(g)(p)$ from φ^*G to F is smooth.

If, in particular, the base M is a point, then G is a Lie group, and we recover the notion of action of a Lie group on a manifold F.

When $F = M$ and φ is the identity, the only action of the Lie groupoid G on M is that given by $S(g)(\alpha g) = \beta g$.

It was originally shown by Ehresmann [**8**] that whenever an action, S, of G on $\varphi : F \to M$ is defined, there is an associated Lie groupoid structure on the manifold φ^*G, with base F. The source map is $(g, p) \mapsto p$, while the target map is $(g, p) \mapsto S(g)(p)$. The partial multiplication is $((g_1, p_1), (g_2, p_2)) \mapsto (g_1 g_2, p_2)$ defined if and only if $g_1 = S(g_2)p_2$, and the inversion is $(g, p) \mapsto (g^{-1}, S(g)(p))$. With this Lie groupoid structure, φ^*G is called the *action groupoid* associated to the given action, and we denote it by $G \lessdot F$. The top arrow in the preceding diagram is a morphism of Lie groupoids from $G \lessdot F$ to G, over $\varphi : F \to M$.

If M is a point, then the Lie groupoid G is a Lie group, and the Lie groupoid structure defined on $G \times F$ reduces to the action groupoid structure described in Section 4 associated to a Lie group action on a manifold.

The Lie functor maps action groupoids to action Lie algebroids. More precisely, the differential, σ, of S is defined by

$$\sigma(X)(p) = TS(p)(X_{\varphi(p)}) ,$$

where $p \in F$ and $X \in \Gamma AG$, while $TS(p)$ denotes the tangent map at the unity, $\varphi(p) \in M$, of the map $g \mapsto S(g)(p)$ from G to F. The result which follows was proved in [**17**].

PROPOSITION 5.2. *Let S be an action of the Lie groupoid G on $\varphi : F \to M$. Then S differentiates to an infinitesimal action of the Lie algebroid AG on F, and the Lie algebroid of the action groupoid $G \ltimes F$ is the action Lie algebroid $AG \ltimes F$.*

Before considering a notion of semi-linear representation for Lie groupoids, we recall the notion of bisection.

DEFINITION 5.3. Let G be a Lie groupoid, with base M, source map α and target map β. A map $b\colon M \to G$ is a *bisection* of G if $\alpha \circ b = \mathrm{Id}_M$ and $\beta \circ b$ is a diffeomorphism of M.

The set of bisections of G is a group under the composition law defined by

$$(b_1 \cdot b_2)(m) = b_1((\beta \circ b_2)(m))b_2(m) ,$$

for bisections b_1 and b_2, and $m \in M$. The *group of bisections* of G is denoted by $\mathcal{B}(G)$.

PROPOSITION 5.4. *Let (E, q, N) be a vector bundle. There is a canonical group isomorphism from $\mathcal{B}(\Phi(E))$ to the group of vector bundle automorphisms of E.*

PROOF. Given a bisection, b, of $\Phi(E)$, we define a bundle map, $\tilde{b}$, from E to itself by

$$\tilde{b}(v) = b(q(v))(v) ,$$

for $v \in E$. Then $\tilde{b}$ is a vector bundle automorphism of E, projecting onto the diffeomorphism $\beta \circ b$ of N. Conversely, given a vector bundle automorphism u of E, the map $x \mapsto u|_{E_x}$, $x \in N$, defines a bisection of $\Phi(E)$. The smoothness of the map is proved by using the local triviality of E. These maps are mutually inverse and they are group morphisms. ∎

It follows from Proposition 4.3 that $\mathcal{B}(\Phi(E))$ is isomorphic to the group of semi-linear isomorphisms of ΓE.

Now consider Lie groupoids G_1 and G_2 on the same base M. It is clear that any Lie groupoid morphism $u\colon G_1 \to G_2$ over M defines a group morphism $U\colon \mathcal{B}(G_1) \to \mathcal{B}(G_2)$ of the corresponding groups of bisections by

$$U(b)(m) = u(b(m)), \tag{23}$$

for $b \in \mathcal{B}(G_1)$ and $m \in M$. Thus a semi-linear representation of a Lie group G on a vector bundle (E, q, M) may be regarded as a group morphism from $\mathcal{B}(G)$ to $\mathcal{B}(\Phi(E))$ which satisfies the smoothness condition in 4.4.

Let G be a Lie groupoid acting on a fibered manifold, $\varphi : F \to M$, with action groupoid $G \ltimes F$. We consider a vector bundle E on base F and a representation of $G \ltimes F$ on E, *i.e.*, a morphism of Lie groupoids from $G \ltimes F$ to $\Phi(E)$. For groupoids in the category of sets, this notion was studied by Brown [**5**]. In the smooth case, such a representation is a global form of a derivative representation. Indeed, by

Proposition 5.2 it differentiates to a representation of $AG \lessdot F$ on E, which by Theorem 3.5 is a derivative representation of AG on E.

Thus for each $g \in G$ and $p \in F_{\alpha g}$, we are given an isomorphism from E_p to $E_{S(g)p}$, where S is the given action of G on F. According to (23) and to Proposition 5.4, there is a group morphism from $\mathcal{B}(G \lessdot F)$ to the group of semi-linear isomorphisms of ΓE. In addition, given a bisection b of G, there is a bisection $\hat{b}$ of $G \lessdot F$ defined by

$$\hat{b}(p) = (b(\varphi(p)), p),$$

for $p \in F$, and the map $b \mapsto \hat{b}$ is a group morphism from $\mathcal{B}(G)$ to $\mathcal{B}(G \lessdot F)$. Composing these maps we obtain a group morphism from $\mathcal{B}(G)$ to the group of semi-linear isomorphisms of ΓE. This group morphism, R, is associated with the given action of G on F, in the sense that if $v \in E_p$ for $p \in F$, then $R(b)v \in E_{p'}$, where $p' = S(b(\varphi p))p$, for all $b \in \mathcal{B}(G)$. Thus we have:

PROPOSITION 5.5. *Given an action of a Lie groupoid G on a fibered manifold $\varphi : F \to M$, to any representation of $G \lessdot F$ on a vector bundle E with base F there corresponds a representation of the group of bisections of G by semi-linear isomorphisms of ΓE, associated to the given action.*

Since the Lie algebra of sections of AG can be formally regarded as the Lie algebra of the infinite-dimensional group $\mathcal{B}(G)$, this proposition can be viewed as a global form of part of Proposition 3.5. When the Lie groupoid G is a Lie group, its group of bisections is the group itself, so this proposition is a partial generalization of Theorem 4.8 to the case of Lie groupoids. We may therefore regard the group morphisms from $\mathcal{B}(G)$ to the group of semi-linear isomorphisms of ΓE satisfying a suitable smoothness condition as the *semi-linear representations* of G on E. However, the characterization of smoothness for such semi-linear representations of groupoids in terms of the bisections of G alone appears to be a difficult matter. The concept of bisection may readily be localized, leading to a sheaf of germs of local bisections, but the relationship between bisections of G and bisections of $G \lessdot F$ evades a simple description.

REMARK 5.6. The representation of the group of bisections in Proposition 5.5 may be approached in an alternative way. We consider a representation of $G \lessdot F$ on E as above and, for $g \in G$ and $p \in F_{\alpha g}$, we denote the isomorphism from E_p to $E_{S(g)p}$ by $\rho(g,p)$. Given $g \in G$, to any section ψ of E over $F_{\alpha g}$ we can associate a section $g \cdot \psi$ of E over $F_{\beta g}$, by

$$(g \cdot \psi)(p) = \rho(g, S(g^{-1})p)(\psi(S(g^{-1})p)), \tag{24}$$

for $p \in F_{\beta g}$. The map $\psi \mapsto g \cdot \psi$ is semi-linear with respect to the action S,

$$(g \cdot (f\psi))(p) = f(S(g^{-1})p)(g \cdot \psi)(p),$$

for $f \in C^\infty(F_{\alpha g})$. By allowing g to range through the values of a bisection, we obtain a semi-linear isomorphism of ΓE: take any $b \in \mathcal{B}(G)$ and for $\psi \in \Gamma E$ and $p \in E$ define

$$(b \cdot \psi)(p) = (g \cdot \psi)(p)$$

where $g = b((\beta \circ b)^{-1}(\varphi(p)))$. It is clear that the resulting map from $\mathcal{B}(G)$ to the group of semi-linear isomorphisms of ΓE is the same as that of Proposition 5.5. This formulation perhaps shows more clearly the effect of the action on sections of

E, but it is still unclear how to characterize the smoothness of ρ in terms of the maps $\psi \mapsto g \cdot \psi$.

6. Appendix: some algebraic formulations

As we remarked in the introduction, both the derivative endomorphisms and the semi-linear isomorphisms are particular cases of the pseudo-linear transformations in the sense of Jacobson. Here, we characterize the pseudo-linear endomorphisms of a left module $\mathcal{E}$ over a ring $\mathcal{C}$ in terms of twisted derivations. In particular, the derivative endomorphisms are characterized in terms of derivations. We also characterize the semi-linear isomorphisms in terms of algebra automorphisms. The case that is relevant to differential geometry is that of $\mathcal{E} = \Gamma E$ and $\mathcal{C} = C^\infty(M)$, where E is a vector bundle over a manifold M. For simplicity, we shall formulate the results in this case only.

The notion of derivation of an algebra can be generalized as follows.

DEFINITION 6.1. Let $\mathcal{A}$ be an algebra and let α be a linear endomorphism of $\mathcal{A}$. A *twisted derivation* of $\mathcal{A}$ with respect to α is a linear endomorphism $\mathcal{U}$ of $\mathcal{A}$ such that, for all $a, b \in \mathcal{A}$,

$$\mathcal{U}(ab) = \mathcal{U}(a)b + \alpha(a)\mathcal{U}(b). \tag{25}$$

It follows from (25) that, if $\mathrm{Im}(\mathcal{U})$ has an element which is not a torsion element, the map α is an algebra endomorphism.

DEFINITION 6.2. An $\mathbb{R}$–linear endomorphism u of ΓE is called *pseudo-linear* if there exist $\mathbb{R}$–linear endomorphisms u^M and u_M of $C^\infty(M)$ such that

$$u(f\psi) = u^M(f)u(\psi) + u_M(f)\psi, \tag{26}$$

for all $f \in C^\infty(M)$ and $\psi \in \Gamma E$.

It follows from (26) that, whenever there exists $\psi \in \Gamma E$ such that $u(\psi)$ and ψ are linearly independent over the ring $C^\infty(M)$, (i) u^M is an algebra endomorphism of $C^\infty(M)$, and (ii) u_M is a twisted derivation of $C^\infty(M)$ with respect to u^M.

Clearly the derivative endomorphisms (resp., the semi-linear isomorphisms) of ΓE are the pseudo-linear endomorphisms for which u^M is the identity of $C^\infty(M)$ (resp., which are bijective and for which $u_M = 0$).

Following Grothendieck [**12**, 16.5], we let $\mathcal{A}(E)$ be the vector space $C^\infty(M) \oplus \Gamma E$ with the commutative $\mathbb{R}$–algebra structure such that the product of any two elements of $C^\infty(M)$ is their product in the ring $C^\infty(M)$, the product of $f \in C^\infty(M)$ and $\psi \in \Gamma E$ is $f\psi$ in the $C^\infty(M)$–module ΓE, and the product of any two sections of ΓE vanishes. We can now formulate the following results, whose proofs are all straightforward computations.

By a slight abuse of language, we let u^M also denote the endomorphism of $\mathcal{A}(E)$ which coincides with u^M on $C^\infty(M)$ and vanishes on ΓE.

PROPOSITION 6.3. *Let $u^M : C^\infty(M) \to C^\infty(M)$, $u_M : C^\infty(M) \to C^\infty(M)$ and $u : \Gamma E \to \Gamma E$ be $\mathbb{R}$–linear maps. Then $u_M + u$ is a twisted derivation of the $\mathbb{R}$–algebra $\mathcal{A}(E)$ with respect to u^M if and only if (i) u_M is a twisted derivation of $C^\infty(M)$ with respect to u^M, and (ii) u is a pseudo-linear endomorphism of ΓE with associated endomorphisms of $C^\infty(M)$, u^M and u_M.*

When u^M is the identity of $C^\infty(M)$, the twisted derivations are ordinary derivations, so we obtain the following corollary.

PROPOSITION 6.4. *Let $u_M : C^\infty(M) \to C^\infty(M)$ and $u : \Gamma E \to \Gamma E$ be $\mathbb{R}$-linear maps. Then $u_M + u$ is a derivation of the $\mathbb{R}$-algebra $\mathcal{A}(E)$ if and only if (i) u_M is a derivation of $C^\infty(M)$, and (ii) u is a derivative endomorphism of ΓE with associated derivation of $C^\infty(M)$, u_M.*

An analogous characterization holds for the semi-linear isomorphisms, in terms of algebra automorphisms.

PROPOSITION 6.5. *Let $u^M : C^\infty(M) \to C^\infty(M)$ and $u : \Gamma E \to \Gamma E$ be $\mathbb{R}$-linear isomorphisms. Then $u^M + u$ is an algebra automorphism of $\mathcal{A}(E)$ if and only if (i) u^M is an algebra automorphism of $C^\infty(M)$, and (ii) u is a semi-linear isomorphism of ΓE with associated automorphism of $C^\infty(M)$, u^M.*

The vector space of pseudo-linear endomorphisms of ΓE is not closed under composition, but it follows from Proposition 6.5 that the semi-linear isomorphisms of ΓE form a group. In addition, just like the set of twisted derivations, the set of pseudo-linear endomorphisms is not closed under commutators, but it follows from Proposition 6.4 that the derivative endomorphisms of ΓE form a Lie algebra.

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Note added in proof

After the submission of the paper, the authors learnt that Rubtsov, in the paper below, also introduced the notion of a *derivation of an A-module over a derivation of an algebra A* in the study of the algebraic counterpart of Lie algebroid cohomology.

V. N. Rubtsov. The cohomology of the Der complex. *Uspekhi Mat. Nauk*, 35(4):209–210, 1980; *Russian Math. Surveys*, 35(4):190-191, 1980.

Centre de Mathématiques, UMR 7640 du CNRS, École Polytechnique, F-91128, Palaiseau, France

E-mail address: yks@math.polytechnique.fr

Department of Pure Mathematics, University of Sheffield, Sheffield, S3 7RH, United Kingdom

E-mail address: K.Mackenzie@sheffield.ac.uk

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Poisson actions and Lie bialgebroid morphisms

Long-guang He, Zhang-Ju Liu, and De-Shou Zhong

ABSTRACT. Poisson actions for Poisson groupoid Γ on Poisson manifold M are characterized by Lie bialgebroid morphisms from T^*M to $\mathcal{A}^*$. When such Lie bialgebroid morphisms can be lifted to Poisson groupoid morphisms $\hat{J}$: $S_M \longrightarrow \Gamma^*$ from the symplectic groupoid of M to the dual Poisson groupoid, we call $\hat{J}$ as a global moment map.

1. Introduction

The purpose of this paper is to study relationships between a Poisson action for Poisson groupoid and the induced infinitesimal action for its tangent Lie bialgebroid and define global moment map.

Fix a Poisson Lie group G with its tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. It is known that (M, π) is Poisson $G-$space if and only if there exists a Lie bialgebroid morphism from T^*M to $\mathfrak{g}^*$[**11**], which is equivalent to : $L_{X_M}\pi = -(d_*X)_M$, $\forall X \in \mathfrak{g}$. We give a similar formula in Lie algebroid action cases and show that such kind of Lie bialgebroid morphisms is really an infinitesimal analogue of the Poisson action for Poisson groupoids. Of course, it also makes sense even if $\mathcal{A}$ is not the tangent Lie algebroid for any Lie groupoid. In order to proceed from local situation to global situation, we should suppose that $\Gamma = \exp(\mathcal{A})$. In general, the image of the exponential map is always an open subgroupoid for any Lie groupoid. We proved that, for a Poisson groupoid (Γ, Ω) with its tangent Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*)$, then a Poisson manifold (M, π) is Poisson $(\Gamma, \Omega)-$space if and only if the vector bundle morphism from T^*M to $\mathcal{A}^*$, the dual of the infinitesimal action map, is Lie bialgebroid morphism.

An important concept introduced in this paper is so called global moment map. When such Lie bialgebroid morphisms as mentioned above can be lifted to Poisson groupoid morphisms $\hat{J} : S_M \longrightarrow \Gamma^*$ from the symplectic groupoid of M to the dual Poisson groupoid, we call $\hat{J}$ as a global moment map, which covers the usual moment map on the base for groupoid actions. In [**6**], Mackenzie proposed a similar approach to moment maps in the Poisson group case, building in a similar way.

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Such Lie groupoid-valued moment maps are generalizations of the group-valued moment maps defined by Lu in [4]. But unlike Poisson group case where one can define equivariant moment map with the dressing action on the dual Poisson Lie group, so far there is no similar notion of dressing action for Poisson groupoids. In [10], Weinstein and Xu defined the concept of global $\mathcal{R}$-matrix, it is just the graph of global moment map lifted from the Lie bialgebra morphism defined by a quasitriangular r-matrix.

Another by-product of our main theorems is that, given an infinitesimal Poisson $\mathcal{A}-$space (M,π) for a coboundary Lie bialgebroid $(\mathcal{A},\mathcal{A}^*,\Lambda)$, if $[\Lambda,\Lambda]_M = 0$, particularly, if Λ is triangular, i.e., $[\Lambda,\Lambda]=0$, then the twisted bivector field $\pi-\Lambda_M$ becomes an $\mathcal{A}$-invariant Poisson structure on M. It would be interesting to compare such kind of twists with the result in [1] about the twist of the symplectic structure for a compact Poisson group action.

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2. Preliminaries

In this section, we recall some basic notions and facts concerning Poisson groupoids and Lie bialgebroids and their actions on manifolds. A smooth manifold M is called Γ-space for a Lie groupoid $(\Gamma \rightrightarrows P,\alpha,\beta)$ means that there are two smooth maps, the moment map and the action map, $J: M \longrightarrow P$ and

$$\sigma : \Gamma \times_J M = \{(x,m) \in \Gamma \times M | \beta(x) = J(m)\} \longrightarrow M$$

such that, writing $\sigma(x,m) = x \cdot m$, the following three conditions:

(1) $J(x \cdot m) = \alpha(x)$,
(2) $(x \cdot y) \cdot m = x \cdot (y \cdot m)$,
(3) $J(m) \cdot m = m$

are satisfied for all compatible $x, y \in \Gamma$ and $m \in M$.

Now suppose that (Γ,Ω) is a Poisson groupoid and Poisson manifold (M,π) is a $\Gamma-$space. The action of Γ on M is Poisson action if its graph $\{(x,m,x\cdot m)|\beta(x) = J(m)\}$ is a coisotropic submanifold of $\Gamma \times M \times \overline{M}$ [9]. Then M is called a Poisson $\Gamma-$space. An equivalent statement for Poisson actions was given in [3] as follows: Note that any bisection K of the groupoid induces a diffeomorphism of Γ by left multiplication by K. This is denoted by l_K. One can also define a diffeomorphism of M by K, denoted by same symbol l_K. Similarly, given any compatible pair $(x,m) \in \Gamma \times_J M$, a local section $\mathcal{Y}$ of the moment map J through the point m induces a map $r_{\mathcal{Y}}$ from a neighborhood of x to a neighborhood of $x \cdot m$.

THEOREM 2.1. [3] With notation above, then (M,π) is Poisson $(\Gamma,\ \Omega)-$space if and only if

(1) For any $f \in C^\infty(P)$,

$$H_{J^*f}(m) = (r_m)_* H_{\alpha^*f}(u), \tag{2.1}$$

where $m \in M$ and $u = J(m)$;

(2) For any compatible $(x,m) \in \Gamma \times_J M$,

$$\pi(x \cdot m) = (l_K)_*\pi(m) + (r_{\mathcal{Y}})_*\Omega(x) - (r_{\mathcal{Y}})_*(l_K)_*\Omega(u), \tag{2.2}$$

where $u = \beta(x) = J(m)$, K is any local bisection of Γ through x, and $\mathcal{Y}$ any local section of J through the point m.

The infinitesimal version of the action of Lie groupoids is the action of Lie algebroids. M is called $\mathcal{A}$–space for a Lie algebroid $\mathcal{A}$ if there is a vector bundle morphism $\Phi : T^*M \longrightarrow \mathcal{A}^*$ and Φ is Poisson map with respect to the Lie-Poisson structures induced from the Lie algebroid structures on their dual bundles respectively. The infinitesimal analogue of Poisson groupoids is the notion of Lie bialgebroids introduced by Mackenzie and Xu in [**7**]. To describe the corresponding infinitesimal Poisson actions for Lie bialgebroids we need to recall firstly some notions and facts on the general vector bundle morphisms and Lie algebroids morphisms.

Let $\mathcal{A}'$ and $\mathcal{A}$ be two vector bundles on bases P' and P, and $\Phi : \mathcal{A}' \longrightarrow \mathcal{A}$ a vector bundle morphism over $J : P' \longrightarrow P$. By $J^!\mathcal{A}$ denote the pull back bundle over P' and for any $A \in \Gamma(\mathcal{A})$ denote $A \circ J$ as corresponding section of $J^!\mathcal{A}$. One can also decompose Φ into two vector bundle morphisms $\Phi^!$ and $J_!$ such that the following commutative diagram holds:

$$\begin{array}{ccccc} \mathcal{A}' & \xrightarrow{\Phi^!} & J^!\mathcal{A} & \xrightarrow{J_!} & \mathcal{A} \\ \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle pr} \\ P' & \xrightarrow{id} & P' & \xrightarrow{J} & P \end{array}$$

where $\Phi^!A' = (\Phi A', x'), \forall A' \in \mathcal{A}'_{x'}$ and $J_!(A, x') = (A, J(x')), \forall A \in \mathcal{A}_{J(x')}$. It is clear that any section of $J^!\mathcal{A}$ can be written in the form $\sum f'_i(A_i \circ J)$ for some $f'_i \in C^\infty(P')$, $A_i \in \Gamma(\mathcal{A})$ and any section A' of $\mathcal{A}'$ has Φ-decompositions as following form:

$$\Phi^! \circ A' = \sum f'_i(A_i \circ J). \tag{2.3}$$

Note that there is a natural vector bundle isomorphism between the dual bundles $(J^!\mathcal{A})^*$ and $J^!\mathcal{A}^*$. and one has the following identity

$$\langle \Phi^*\xi, A' \rangle = \langle \xi \circ J, \Phi^!A' \rangle, \ \forall \xi \in \Gamma(\mathcal{A}^*), A' \in \Gamma(\mathcal{A}'). \tag{2.4}$$

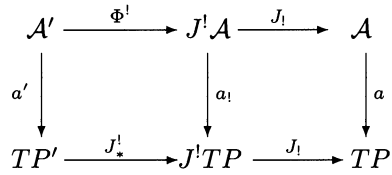
DEFINITION 2.2. [**2**] Let $\mathcal{A}', \mathcal{A}$ be Lie algebroids on bases P', P with anchors a', a respectively. Then a morphism of Lie algebroids from $\mathcal{A}'$ to $\mathcal{A}$ is a vector bundle morphism $\Phi : \mathcal{A}' \longrightarrow \mathcal{A}$ over $J : P' \longrightarrow P$ such that

(1) $a \circ \Phi = J_* \circ a'$;
(2) $\forall A', B' \in \Gamma(\mathcal{A}')$ with Φ-decompositions:

$$\Phi^! \circ A' = \sum u'_i(A_i \circ J), \quad \Phi^! \circ B' = \sum v'_j(B_j \circ J),$$

$$\Phi^![A', B'] = \sum u'_i v'_j([A_i, B_j] \circ J) + \sum [A', v'_j](B_j \circ J) - \sum [B', u'_i](A_i \circ J). \tag{2.5}$$

It is clear that condition (1) in above definition is equivalent to the following commutative diagram:



where the induced anchor $a_!$ from $J^!\mathcal{A}$ to $J^!TP$ is defined by $a_!(A, x') = (a(A), x')$, for all $A \in \mathcal{A}_{J(x')}, x' \in P'$.

For two Lie bialgebroids $(\mathcal{A}, \mathcal{A}^*)$ and $(\mathcal{B}, \mathcal{B}^*)$, according to [7], a vector bundle morphism $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ is called a morphism of Lie bialgebroids if it is both a Lie algebroid morphism and a Poisson map with respect to the Lie-Poisson structures induced from the Lie algebroid structures on their dual bundles respectively.

3. Main Results

In this section, we state our main results in this paper and put their proofs in the next section. We also show some related properties, examples and introduce the concept of global moment map.

THEOREM 3.1. Let $\mathcal{A}^*$ be a Lie algebroid over P with the anchor ρ_* and (M, π) a Poisson manifold. Then, for the cotangent Lie algebroid T^*M induced from the Poisson structure, a vector bundle morphism $\Phi : T^*M \longrightarrow \mathcal{A}^*$ over $J : M \longrightarrow P$ is Lie algebroid morphism if and only if the following two conditions hold:

(1) $H_{J^*f} = -(d_*f)_M$, $\forall f \in C^\infty(P)$;
(2) $L_{X_M}\pi = -(d_*X)_M$, $\forall X \in \Gamma(\mathcal{A})$,

where H_{J^*f} denotes Hamiltonian vector field on M and $X_M = \Phi^*(X)$. The differential d_* on $\Gamma(\Lambda^*\mathcal{A})$ comes from the Lie algebroid structure on $\mathcal{A}^*$.

For a Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*)$, suppose that M is a $\mathcal{A}-$space. Then the vector bundle morphism $\Phi : T^*M \longrightarrow \mathcal{A}^*$, the dual of the action map, is Poisson map just by definition. So that if Φ is a Lie algebroid morphism Φ is also a Lie bialgebroid morphism automatically.

DEFINITION 3.2. A Poisson manifold (M, π) is called an infinitesimal Poisson $\mathcal{A}-$space for Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*)$ if there exists a Lie bialgebroid morphism from T^*M to $\mathcal{A}^*$.

A natural example is that, for any Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*)$ over M, $\rho^* : T^*M \longrightarrow \mathcal{A}^*$ is a Lie bialgebroid morphism with moment map $J = id_M$, where ρ is the anchor of $\mathcal{A}$ and M with the induced Poisson structure from the Lie bialgebroid structure.

For a Poisson Lie group G with its tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, it is known that (M, π) is Poisson $G-$space if and only if there exists a Lie bialgebroid morphism from T^*M to $\mathfrak{g}^*$[11]. The next theorem shows that such a kind of Lie bialgebroid morphism is really an infinitesimal analogue of the Poisson action for Poisson groupoids. Of course, it also makes sense even if $\mathcal{A}$ is not the tangent Lie algebroid for any Lie groupoid. In order to proceed from local situation to global situation, we should suppose that $\Gamma = \exp(\mathcal{A})$. In general the image of the exponential map is always an open subgroupoid for any Lie groupoid.

THEOREM 3.3. Let (Γ, Ω) be a Poisson groupoid with its tangent Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*)$. Then a Poisson manifold (M, π) is a Poisson $(\Gamma, \Omega)-$space if and only if

the vector bundle morphism from T^*M to $\mathcal{A}^*$, the dual of the infinitesimal action map, is Lie bialgebroid morphism

In [**11**] Xu proved that, for any $A \in \Gamma(\mathcal{A})$, $d_*A = 0$ iff $\exp tA$ are coisotropic submanifolds in the Poisson groupoid Γ for all t for which they exist. Combining his result with (2) in theorem 3.1 and theorem 3.3, we have

COROLLARY 3.4. Let $K = \exp tA$ be a coisotropic bisection of Poisson groupoid Γ, then its induced diffeomorphism l_K on a Poisson $\Gamma-$space M preserves the Poisson structure of M.

It is known that the moment map is a Poisson map for any Poisson $\Gamma-$space. This fact is also true for its infinitesimal version.

COROLLARY 3.5. The moment map of any infinitesimal Poisson $\mathcal{A}-$space for Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*)$ is Poisson map.

PROOF. It is easy to see that condition (1) in theorem 3.1 with the fact Φ being Poisson means that J is the Poisson map. □

DEFINITION 3.6. Let (Γ, Ω) be a Poisson groupoid with its tangent Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*)$ and (M, π) a Poisson $(\Gamma, \Omega)-$space, If the Lie bialgebroid morphism $\Phi : T^*M \longrightarrow \mathcal{A}^*$ over $J : M \longrightarrow P$ can be lifted to a Poisson groupoid morphism $\hat{J} : S_M \longrightarrow \Gamma^*$ from the symplectic groupoid of M to the dual Poisson groupoid (in case that both of them exist), then $\hat{J}$ is called the global moment map.

REMARK 3.1. (1). By same way, one can define the global moment map for an infinitesimal Poisson $\mathcal{A}-$space for Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*)$. (2). Such Lie groupoid-valued moment maps are generalizations of the group-valued moment maps defined by Lu in [**4**]. Even if the action of Γ can be lifted to S_M, unlike Poisson group case where one can define equivariant moment map for the dressing action, so far there is no similar notion of dressing action for Poisson groupoids. (3). In [**10**], Weinstein and Xu defined the concept of global $\mathcal{R}$-matrix, it is just the graph of global moment map lifted from the Lie bialgebra morphism defined by a quasitriangular r-matrix.

Another special case was discussed in [**8**], where an example is follows. Let G be a Lie group and suppose that symplectic manifold (M, ω) is a Hamiltonian G-space with an Ad-equivariant moment map $\varphi : M \longrightarrow \mathfrak{g}^*$. Thus, $S_M = M \times M$ with symplectic structure $(\omega, -\omega)$, and one can take the global moment map as:

$$\hat{J} : M \times M \longrightarrow \mathfrak{g}^*, \ \hat{J}(x, y) = \varphi(x) - \varphi(y).$$

It is easy to see that $\hat{J}$ is Poisson groupoid morphism. The usual defined moment map for Poisson groupoid actions is zero in this case, which is the restriction of $\hat{J}$ on the diagonal.

Next let $(\mathcal{A}, \mathcal{A}^*, \Lambda)$ be a coboundary Lie bialgebroid for a bivector field $\Lambda \in \Gamma(\wedge^2\mathcal{A})$. The induced differential on $\Gamma(\wedge^\bullet\mathcal{A})$ has the form, $d_* = [\Lambda, \cdot\,]$. For this situation, we have

COROLLARY 3.7. Let (M, π) is an infinitesimal Poisson $\mathcal{A}-$space for a coboundary Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*, \Lambda)$. If $[\Lambda, \Lambda]_M = 0$, particularly, if Λ is triangular, i.e., $[\Lambda, \Lambda] = 0$, then the twisted bivector field $\pi - \Lambda_M$ becomes an $\mathcal{A}$-invariant Poisson structure on M.

PROOF. By formula (2) in theorem 3.1,

$$L_{X_M}\pi = -(d_*X)_M = -[\Lambda, X]_M = -[\Lambda_M, X_M],\ \forall X \in \Gamma(A).$$

That is,

$$L_{X_M}(\pi - \Lambda_M) = 0,\ \forall X \in \Gamma(A),$$

so that $\pi - \Lambda_M$ is an $\mathcal{A}$-invariant bivector field on M. This means that

$$[\Lambda_M, \pi - \Lambda_M] = 0$$

as well as

$$[\pi - \Lambda_M, \pi - \Lambda_M] = -2[\pi, \Lambda_M] + [\Lambda_M, \Lambda_M] = -[\Lambda, \Lambda]_M.$$

That is, $\pi - \Lambda_M$ is Poisson structure iff $[\Lambda, \Lambda]_M = 0$. □

It would be interesting to compare such kind of twists with the result in [1] about the twist of the symplectic structure for a compact Poisson group action.

We end this section with an example. For complete Poisson group G with dual Poisson group G^*, the symplectic groupoid of G^* is $\Gamma = GG^*$([5]). Suppose that (M, π) is a Poisson $G-$space with an equivariant moment map $J : M \longrightarrow G^*$. Then it is known that (M, J) is a Poisson $\Gamma-$space under the $\Gamma-$ action

$$(a, u) \cdot x = a \cdot x, \quad a \in G, \quad u \in G^* \text{ and } \ x \in M \text{ such that } \ J(x) = u.$$

The infinitesimal version for this example is well - stated for the case that there is no global action on M and no global symplectic groupoid of G^*. Suppose that $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra and (M, π) is an infinitesimal Poisson $\mathfrak{g}-$ space with an equivariant moment map $J : M \longrightarrow G^*$. Take $\mathcal{A} = T^*G^* \cong G^* \times \mathfrak{g}$ and $\mathcal{A}^* = TG^* \cong G^* \times \mathfrak{g}^*$. Then the vector bundle morphism $\phi : T^*M \longrightarrow \mathcal{A}^*$ is defined by

$$\langle \phi(\alpha), A\rangle(x) = \langle A_M(x), \alpha\rangle,\ \forall \alpha \in T^*_xM,\ A \in \mathfrak{g} \cong \mathcal{A}_{J(x)},\ x \in M.$$

Condition (1) in theorem 3.1 is satisfied because J is Poisson map. Condition (2) is satisfied for all constant sections of T^*G^*, i.e., elements of $\mathfrak{g}$ obviously so that from the fact that any section of T^*G^* can be represented as a linear combination of fA for some $f \in C^\infty(G^*)$ and $A \in \mathfrak{g}$. Thus ϕ is Lie bialgebroid morphism. That is, (M, π) is an infinitesimal Poisson $\mathcal{A}-$space for Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*)$. If the global moment map exists, it should be a Poisson groupoid morphism from S_M to $G^* \times \overline{G^*}$ over J.

Next we begin to prove the two theorems given in this section.

4. Proof of main theorems

PROOF. (Theorem 3.1) It is easy to see that

$$\begin{aligned} J_* \circ \pi^\# = \rho_* \circ \Phi \quad &\Longleftrightarrow \quad -\pi^\# \circ J^* = \Phi^* \circ \rho_*^* \\ &\Longleftrightarrow \quad -\pi^\# \circ J^*\delta f = \Phi^* \circ \rho_*^* \delta f,\ \forall f \in C^\infty(P) \\ &\Longleftrightarrow \quad H_{J^*f} = -\Phi^*(d_*f) = -(d_*f)_M, \end{aligned}$$

where δ is the usual exterior differential on manifolds and $\rho_*^*\delta f = d_*f$[7]. That is, formula (1) in the theorem is equivalent to formula (1) in definition 2.2. Thus now we can suppose the corresponding commutative diagram shown in the last section is satisfied, i.e.,

$$\begin{array}{ccccc} T^*M & \xrightarrow{\Phi^!} & J^!\mathcal{A}^* & \xrightarrow{J_!} & \mathcal{A}^* \\ \downarrow{\scriptstyle \pi^\#} & & \downarrow{\scriptstyle \rho_*!} & & \downarrow{\scriptstyle \rho_*} \\ TM & \xrightarrow{J^!_*} & J^!TP & \xrightarrow{J_!} & TP \end{array}$$

Next we will figure out the second formula in the theorem. For any $\alpha, \beta \in \Gamma(T^*M)$ with Φ-decompositions shown in definition 2.2,

$$\Phi^!\alpha = u^i(\xi_i \circ J), \quad \Phi^!\beta = v^j(\eta_j \circ J),$$

where $\xi_i,\ \eta_j \in \Gamma(\mathcal{A}^*)$ and u^i, v^j are smooth functions on M (it is sufficient to consider only generating elements). From above picture, it is easy to check the following equality

$$\langle \pi^\#(\alpha), J^*\omega\rangle = u^i\langle \rho_*(\xi_i), \omega\rangle \circ J,\ \forall \omega \in \Omega^1(P). \tag{4.1}$$

Now let $X \in \Gamma(\mathcal{A})$ and set

$$K = \Phi^![\alpha,\beta]_\pi - \{u^iv^j[\xi_i,\eta_j] \circ J + \pi^\#\alpha(v^j)\eta_j \circ J - \pi^\#\beta(u^i)\xi_i \circ J\} \ \in \Gamma(J^!\mathcal{A}^*),$$

we have

$$\begin{aligned}
\langle (d_*X)_M, \alpha \wedge \beta\rangle &= \langle d_*X \circ J, \Phi^!\alpha \wedge \Phi^!\beta\rangle \\
&= \langle d_*X \circ J, u^i(\xi_i \circ J) \wedge v^j(\eta_j \circ J)\rangle \\
&= u^iv^j\langle d_*X, \xi_i \wedge \eta_j\rangle \circ J \\
&= u^iv^j\{\langle \rho_*(\xi_i), \delta\langle X, \eta_j\rangle\rangle - \langle \rho_*(\eta_j), \delta\langle X, \xi_i\rangle\rangle - \langle X, [\xi_i, \eta_j]\rangle\} \circ J \\
&= v^j\langle \pi^\#(\alpha), J^*\delta\langle X, \eta_j\rangle\rangle - u^i\langle \pi^\#(\beta), J^*\delta\langle X, \xi_i\rangle\rangle - \langle X \circ J, u^iv^j[\xi_i, \eta_j]\rangle \\
&= v^j\pi^\#(\alpha)(\langle X, \eta_j\rangle \circ J) - u^i\pi^\#(\beta)(\langle X, \xi_i\rangle \circ J) - \langle X \circ J, \Phi^![\alpha,\beta]_\pi\rangle \\
&\quad + \langle X \circ J, \pi^\#\alpha(v^j)(\eta_j \circ J)\rangle - \langle X \circ J, \pi^\#\beta(u^i)(\xi_i \circ J)\rangle + \langle X \circ J, K\rangle \\
&= \pi^\#\alpha\langle X \circ J, \Phi^!\beta\rangle - \pi^\#\beta\langle X \circ J, \Phi^!\alpha\rangle - \langle X_M, [\alpha,\beta]_\pi\rangle + \langle X \circ J, K\rangle \\
&= \pi^\#\alpha\langle X_M, \beta\rangle - \pi^\#\beta\langle X_M, \alpha\rangle - \langle X_M, [\alpha,\beta]_\pi\rangle + \langle X \circ J, K\rangle \\
&= \langle [\pi^\#\alpha, X_M], \beta\rangle + \langle X_M, L_{\pi^\#\alpha}\beta\rangle - \langle [\pi^\#\beta, X_M], \alpha\rangle \\
&\quad - \langle X_M, L_{\pi^\#\beta}\alpha\rangle - \langle X_M, L_{\pi^\#\alpha}\beta - L_{\pi^\#\beta}\alpha - d\pi(\alpha,\beta)\rangle + \langle X \circ J, K\rangle \\
&= -X_M(\pi^\#(\alpha,,\beta)) + \langle \pi^\#\alpha, L_{X_M}\beta\rangle + X_M(\pi^\#(\beta,\alpha)) \\
&\quad - \langle \pi^\#\beta, L_{X_M}\alpha\rangle + X_M(\pi^\#(\alpha,\beta)) \\
&= \pi(\alpha, L_{X_M}\beta) + \pi(L_{X_M}\alpha, \beta) - X_M(\pi(\alpha,\beta)) + \langle X \circ J, K\rangle \\
&= -(L_{X_M}\pi)(\alpha,\beta) + \langle X \circ J, K\rangle.
\end{aligned}$$

In above computation, formulas 2.4 and 4.1 are used in step 2 and step 6 respectively. We see that formula (2) in Theorem 3.1 holds is equivalent to $K = 0$, i.e., Φ is Lie algebroid morphism by definition 2.2. Thus the proof is finished. □

PROOF. (Theorem 3.3) Let $(\mathcal{A}, \mathcal{A}^*)$ be the tangent Lie bialgebroid of the Poisson groupoid (Γ, Ω) over P. Poisson manifold (M, π) is a $\Gamma-$space with the moment map J. When consider the left groupoid action, one always takes $\mathcal{A} = \bigcup_{u\in P} T_u\beta^{-1}(u)$ and $\rho = \alpha_*$ as the anchor of $\mathcal{A}$. As the first step, we prove that two condition (1) in Theorems 2.1 and 3.1 are equivalent. By Proposition 4.4.5 in [**9**],

$$d_*f = \rho_*^*(\delta f) = -\Omega^{\#}(\delta\alpha^* f) = -H_{\alpha^* f},\ \forall f \in C^\infty(P).$$

From this fact, it is easy to see that condition (1) in Theorem 2.1 is equivalent to condition (1) in Theorem 3.1.

By our assumption, for any $x \in \Gamma$, there exist some t_0 and $X \in \Gamma(\mathcal{A})$ such that $\mathcal{K} = \exp t_0 X$ as a local bisection through x. Let $\beta(x) = J(m) = u \in P$ and $\mathcal{Y}$ a local section of J through m. In addition, taking

$$m' = \exp t_0 X \cdot m,\ \ u' = \alpha(\exp t_0 X(u)) = \alpha(x). \tag{4.2}$$

Obviously, $(u', m') \in \Gamma \times_J M$. and $l_{\exp t_0 X}\mathcal{Y} = \mathcal{Y}'$ is a local section of J through m'.

With notation above we first suppose that the action of Γ on M is Poisson, By Theorem 2.1, we have

$$\pi(\exp tX \cdot m) = (l_{\exp tX})_*\pi(m) + (r_{\mathcal{Y}})_*\Omega(\exp tX \cdot u) - (r_{\mathcal{Y}})_*(l_{\exp tX})_*\Omega(u).$$

That is,

$$(l_{\exp -tX})_*\pi(\exp tX \cdot m) = \pi(m) + (r_{\mathcal{Y}})_*(l_{\exp -tX})_*\Omega(\exp tX \cdot u) - (r_{\mathcal{Y}})_*\Omega(u).$$

Taking derivative at $t = 0$, we obtain an equality

$$L_{X_M}\pi(m) = (r_{\mathcal{Y}})_*(L_{X^r}\Omega(u)), \tag{4.3}$$

where X^r denotes the right-invariant vector field on Γ generated by X. Combining the fact that ([**11**])

$$L_{X^r}\Omega = [X^r, \Omega] = -(d_*X)^r$$

we have

$$L_{X_M}\pi = -(r_{\mathcal{Y}})_*(d_*X) = -(d_*X)_M.$$

This is just formula (2) in Theorem 3.1. Thus, we proved one direction of the theorem.

Conversely, suppose that (M, π) is an infinitesimal Poisson $\mathcal{A}$–space. That is, equality 4.3 holds. Define a family of bivector fields $\mathcal{F}(t, \cdot)$ on M as follows, $\forall m \in M,\ \forall X \in \Gamma(\mathcal{A})$,

$$\mathcal{F}(t, m) = (l_{\exp -tX})_*\pi(\exp tX \cdot m) - (r_{\mathcal{Y}})_*(l_{\exp -tX})_*\Omega(\exp tX \cdot u) \in \wedge^2 T_m M.$$

By the definition of Lie derivative and 4.3, we have

$$\frac{d}{dt}\mathcal{F}(0, m) = L_{X_M}\pi(m) - (r_{\mathcal{Y}})_* L_{X^r}\Omega(u) = 0 \tag{4.4}$$

Moreover, since Γ is a Poisson groupoid, one has

$$\Omega(xy) = (l_{\mathcal{X}})_*\Omega(y) + (r_{\mathcal{Y}})_*\Omega(x) - (r_{\mathcal{Y}})_*(l_{\mathcal{X}})_*\Omega(u),$$

where $x, y \in \Gamma$ are compatible and $u = \beta(x)$, $\mathcal{X}, \mathcal{Y}$ are local bisections through x and y respectively. By mean of this formula, we see that

$$\begin{aligned}
\mathcal{F}(t+t_0, m) &= (l_{e^{-(t+t_0)x}})_*\pi(e^{(t+t_0)X}\cdot m) - (r_{\mathcal{Y}})_*(l_{e^{-(t+t_0)x}})_*\Omega(e^{(t+t_0)X}\cdot u) \\
&= (l_{e^{-t_0x}})_*\{(l_{e^{-tx}})_*\pi(e^{tX}\cdot m') - (r_{\mathcal{Y}})_*(l_{e^{-tx}})_*\Omega(e^{tX}\cdot e^{t_0X}(u))\} \\
&= (l_{e^{-t_0x}})_*\{(l_{e^{-tx}})_*\pi(e^{tX}\cdot m') - (r_{\mathcal{Y}})_*(l_{e^{-tx}})_*[(l_{e^{tx}})_*\Omega(e^{t_0X}(u)) \\
&\quad + (r_{e^{t_0x}})_*\Omega(e^{tX}(u')) - (l_{e^{tx}})_*(r_{e^{t_0x}})_*\Omega(u)]\} \\
&= (l_{e^{-t_0x}})_*\{(l_{e^{-tx}})_*\pi(e^{tX}\cdot m') - (r_{\mathcal{Y}})_*[\Omega(e^{t_0X}(u)) \\
&\quad + (l_{e^{-tx}})_*(r_{e^{t_0x}})_*\Omega(e^{tX}(u')) - (r_{e^{t_0x}})_*\Omega(u)]\},
\end{aligned}$$

where m' and u' are defined in 4.2. Consequently,

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}(t_0, m) &= (l_{\exp -t_0X})_*\{L_{X_M}\pi(m') - (r'_{\mathcal{Y}})_*L_{X^r}\Omega(u'))\} \\
&= (l_{\exp -t_0X})_*\frac{d}{dt}\mathcal{F}(0, m') \\
&= 0
\end{aligned}$$

Therefore, $\mathcal{F}(t, m) = \mathcal{F}(0, m)$ for all t. From the fact that

$$\mathcal{F}(0) = \pi(m) - (r_{\mathcal{Y}})_*\Omega(u)$$

we have

$$(l_{\exp -tX})_*\pi(\exp tX\cdot m) - (r_{\mathcal{Y}})_*(l_{\exp -tX})_*\Omega(\exp tX\cdot u) = \pi(m) - (r_{\mathcal{Y}})_*\Omega(u)$$

Set $t = t_0$, then we get

$$\pi(x\cdot m) = (l_{\mathcal{K}})_*\pi(m) + (r_{\mathcal{Y}})_*\Omega(x) - (r_{\mathcal{Y}})_*(l_{\mathcal{K}})_*\Omega(u).$$

This is just formula (2) in Theorem 2.1. So we complete our proof. □

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Department of Mathematics, Capital Normal University,Beijing,100037, China
E-mail address: helg@mail.cnu.edu.cu

LPAM and Department of Mathematics, Peking University, Beijing,100871,China
E-mail address: liuzj@pku.edu.cn

Institute of Mathematics, Peking University, Beijing,100871,China
E-mail address: Zhongdsh@X263.net

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Identities and derivations for Jacobian algebras

A. S. Dzhumadil'daev

ABSTRACT. Constructions of n-Lie algebras from strong n-Lie-Poisson algebras are given. First cohomology groups of adjoint module of the Jacobian algebras are calculated. Minimal identities of 3-Jacobian algebra are found.

1. Introduction

Let U be an associative commutative algebra over a field K with commuting derivations $\partial_1, \ldots, \partial_n$. Say, U is the algebra of differentiable functions on an n-dimensional manifold or the polynomial algebra $K_n^+ = K[x_1, \ldots, x_n]$ or the Laurent polynomial algebra $K_n = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. In these examples, $\partial_i = \partial/\partial x_i$ are partial derivations. If not stated otherwise, the characteristic p of the field K is supposed to be 0.

Let $Jac_n^S : \wedge^n U \to U$ be the Jacobian map:

$$Jac_n^S(u_1, \ldots, u_n) = \det(\partial_i u_j) = \begin{vmatrix} \partial_1 u_1 & \cdots & \partial_1 u_n \\ \cdots & \cdots & \cdots \\ \partial_n u_1 & \cdots & \partial_n u_n \end{vmatrix}$$

Define $(n+1)$-linear map $Jac_{n+1}^W : \wedge^{n+1} U \to U$ by

$$Jac_{n+1}^W(u_0, u_1, \ldots, u_n) = \begin{vmatrix} u_0 & u_1 & \cdots & u_n \\ \partial_1 u_0 & \partial_1 u_1 & \cdots & \partial_1 u_n \\ \cdots & \cdots & \cdots & \cdots \\ \partial_n u_0 & \partial_n u_1 & \cdots & \partial_n u_n \end{vmatrix}$$

In terms of wedge products we see that

$$Jac_n^S = \partial_1 \wedge \cdots \wedge \partial_n,$$
$$Jac_{n+1}^W = id \wedge \partial_1 \cdots \wedge \partial_n,$$

where $id : U \to U$, $u \mapsto u$, is the identity map.

In [**5**], [**6**], [**2**] it was proved that the n-ary operation $\omega = Jac_n^S$ satisfies the identity

$$g^\omega = 0, \tag{1}$$

where

$$g^\omega(u_1, \ldots, u_{2n-1}) = \omega(u_1, \ldots, u_{n-1}, \omega(u_n, \ldots, u_{2n-1})) -$$

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$$\sum_{i=1}^{n} \omega(u_1, \dots, u_{i-1}, \omega(u_1, \dots, u_{n-1}, u_i), u_{i+1}, \dots, u_{2n-1})$$

In [9] this identity is called the *Fundamental Identity.* We call it the *Fundamental Identity of type* I.

We notice that the Jacobian $\omega = Jac_n^S$ satisfies one more identity, which we call the *Fundamental Identity of type* II:

$$f^{\omega} = 0, \tag{2}$$

where

$$f^{\omega}(u_1, \dots, u_{n-1}, v_1, \dots, v_{n+1}) =$$
$$\sum_{i=1}^{n+1} (-1)^i \omega(u_1, \dots, u_{n-1}, v_i) \cdot \omega(v_1, \dots, \hat{v_i}, \dots, v_{n+1}).$$

It is known that the Jacobian $\omega = Jac_n^S$ satisfies the Leibniz identity for the multiplication $\cdot$:

$$\omega(u \cdot u', u_2, \dots, u_n) = u \cdot \omega(u', u_2, \dots, u_n) + u' \cdot \omega(u, u_2, \dots, u_n). \tag{3}$$

Here $u, u', u_1, \dots, u_{2n+1}, v_1, \dots, v_{n+1}$ are any elements of U.

In our paper we consider algebras with many operations. An operation or multiplication on an algebra is a polylinear map. If $\omega : U \times \cdots \times U \to U$ is a polylinear map with n arguments, then ω is called a *n-ary multiplication* on U. The space of n-ary polylinear maps on U is denoted by $T^n(U, U)$. If $n = 0$, then we set $T^0(U, U) = U$. The set of operations on U is called *a signature* of algebra [7]. The algebra with the vector space U and signature $\Omega = \{\omega, \eta, \dots\}$ is denoted as (U, Ω) or $(U, \omega, \eta, \dots)$ or just U, when it is clear which multiplications are considered.

An algebra (U, ω) with skew-symmetric n-ary multiplication ω that satisfies (1) is called *n-Lie* [5]. An algebra $(U, \cdot, \omega)$ is called *n-Lie-Poisson* if $(U, \cdot)$ is an associative commutative algebra and it satisfies the identities (1) and (3). If it satisfies one more identity, namely the identity (2), then this algebra is called *strong n-Lie-Poisson.*

Sometimes n-Lie algebras are called Nambu [8], Filipov or Takhtajan algebras and n-Lie-Poisson algebras are called Nambu-Poisson algebras.

QUESTION. Does the fundamental identity of type II follow from the fundamental identity of type I and from the Leibniz identity, if $n > 2$ and $p = 0$?

Here we suppose that the commutativity and associativity identities for the binary multiplication and the skew-symmetric identity for n-multiplication are given. Otherwise, more likely, the answer to this question would be negative.

For some statement $\mathcal{X}$ denote by $\delta(\mathcal{X})$ its Kroneker symbol: $\delta(\mathcal{X}) = 1$ if $\mathcal{X}$ is true and $\delta(\mathcal{X}) = 0$ if $\mathcal{X}$ is false. For a set of vectors Y denote by $< Y >$ its linear span.

Let

$$r^{\omega}(u_1, \dots, u_{n-2}, u_{n-1}, u_n, \dots, u_{2n}) =$$
$$\sum_{i=n-1, n+1, \dots, 2n} (-1)^{i+n+\delta(i \le n-1)} \omega(u_1, u_2, \dots, u_{n-2}, u_i) \omega(u_n, u_{n+1}, \dots, \hat{u_i}, \dots, u_{2n}) +$$
$$\sum_{i=n-1, n+1, \dots, 2n} (-1)^{i+n+\delta(i \le n-1)} \omega(u_n, u_2, \dots, u_{n-2}, u_i) \omega(u_1, u_{n+1}, \dots, \hat{u_i}, \dots, u_{2n}).$$

So, r^ω is the polynomial with n skew-symmetric arguments $u_{n-1}, u_{n+1}, \ldots, u_{2n}$ and two symmetric arguments u_1 and u_n.

By theorem 6 of [**2**] the answer to our question is positive if the inverse of this statement is true: the identity $f^\omega = 0$ follows from the identity $r^\omega = 0$ and the Leibniz rule (3).

We give a positive answer to the last statement for $n = 3$ (theorem 5.1). If $p > 0$, it is more likely that the answer would be negative. For $n = 2$ the answer is also negative. There exist examples of Lie-Poisson algebras that are not strong. For example, $(K[x_1, x_2, x_3, x_4], \partial_1 \wedge \partial_2 + \partial_3 \wedge \partial_4)$ is such an algebra.

Let (A, Ω) be an algebra with some vector space A and signature Ω. For $\Omega' \subseteq \Omega$ a linear map $D : A \to A$ is called Ω'-derivation, if

$$D(\omega(a_1, \ldots, a_n)) = \sum_{i=1}^{n} \omega(a_1, \ldots, a_{i-1}, D(a_i), \ldots, a_n),$$

for any $a_1, \ldots, a_n \in A$ and for all $\omega \in \Omega'$. Call Ω-derivation a derivation. Let $Der\,(A, \Omega')$ be a space of all derivations of (A, Ω'). Set $Der\,A = Der\,(A, \Omega)$.

In terms of the operators $L_{a_1,\ldots,a_{n-1}} : A \to A, a \mapsto \omega(a_1, \ldots, a_{n-1}, a)$ we see that ω satisfies (1) if and only if

$$L_{a_1,\ldots,a_{n-1}} \in Der A,$$

for any $a_1, \ldots, a_{n-1} \in A$. Derivations of the form $L_{a_1,\ldots,a_{n-1}}$ are called *interior* derivations. Let $Int\,A$ be a space of interior derivations. If A is n-Lie, then $Int\,A$ is a Lie algebra under the commutator of operators. Moreover, $Int\,A$ is an ideal of $Der\,A$:

$$[D, L_{a_1,\ldots,a_{n-1}}] = \sum_{i=1}^{n} L_{a_1,\ldots,a_{i-1},D(a_i),a_{i+1},\ldots,a_n},$$

for any $D \in Der\,A, a_1, \ldots, a_n \in A$. In particular one can consider a factor-algebra, the algebra of outer derivations, $Out\,A = Der\,A/Int\,A$.

In [**5**] two examples of n-Lie algebras were constructed. The first one is the vector products algebra and the second one is the Jacobian algebra. In this paper it was established that any derivation of the vector products algebra is interior. In [**5**] it was also noticed that, if (A, ω) is $(n+1)$-Lie, then $(A, i(a)\omega)$ is n-Lie, where n-ary map $i(a)\omega$ is defined by

$$i(a)\omega(a_1, \ldots, a_n) = \omega(a, a_1, \ldots, a_n).$$

In our paper we give a generalisation of the Jacobian algebras. Namely, we establish that the algebra (U, Jac^W_{n+1}) becomes a $(n+1)$-Lie algebra. If U has a unit element 1, then $\partial_i(1) = 1$, for any $i = 1, \ldots, n$. If U has a unit, then the $(n+1)$-ary algebra (U, Jac^W_{n+1}) allows us to obtain the n-ary Jacobian algebra (U, Jac^S_n) by the restriction operation: $Jac^S_n = i(1)Jac^W_{n+1}$.

The theory of polynomial identities is well developed for ordinary algebras, i.e., for algebras with binary operations. The case of multi operation algebras needs some detailed information about n- or Ω-words and Ω-polynomials. Necessary definitions and descriptions of Ω-words are given in section 2 (theorem 2.1). We introduce a notion of Ω-degree for Ω-polynomial, that is, the number of operations. For example, the Jacobian algebra considered as an n-Lie algebra, has only one n-ary operation, denoted by $\mu_n = Jac^S_n$, and it has only one identity of Ω-degree 1

(skew-symmetric identity for the Jacobian) and only one identity of Ω-degree 2 (n-Lie identity). The identity $f^{\mu_n} = 0$ is not an n-Lie identity, since the construction of f^{ω} needs binary operation. Probably Jacobian algebra as n-Lie algebra has no any identity of Ω-degree 3. If we consider the Jacobian algebras as n-Lie-Poisson algebras, i.e., as algebras with one binary operation $\mu_2 : (u, v) \mapsto u \cdot v$ and one n-ary operation $\mu_n = Jac_n^S$, then it has two identities of Ω-degree 1 (commutativity for μ_2 and skew-symmetry for μ_n) and three identities of Ω-degree 2 (n-Lie for μ_n, Leibniz rule between μ_2 and μ_n, and associativity identity for μ_2). As n-Lie-Poisson algebra, the Jacobian algebra has one more identity of Ω-degree 3 (identity between two μ_n-s and one μ_2). It seems that both directions of studying Ω-identities of the Jacobian algebras will be very interesting. In our paper we describe Ω-degree 2 identities for the Jacobian algebra (K_3, Jac_3^S) as 3-Lie algebra.

In cocycle and identity constructions we use two methods: a polynomial principle and $\mathcal{D}$-invariants method (section 7). The polynomial principle allows us to restrict our considerations by the case of polynomial algebras. As it turns out, almost all of our cocycles and polynomials are $\mathcal{D}$-invariant. To prove that $\mathcal{D}$-invariant polynomial is an identity it is sufficient to calculate these polynomials on their supports. In other words, the identity (cocyclicity) checking problem we reduce to the calculation problem whether a polynomial is equal to 0 on some concrete arguments. Here the use of computer calculations is very helpful.

Let $U = K_n$. In section 8 we prove that the classes of the following linear maps consist of the basis of $Out\,(U, Jac_n^S)$:

$$\Delta := \sum_{i=1}^{n} x_i \partial_i + n(1-n)^{-1},$$

$$D_{-\theta} : x^{\alpha} \mapsto \delta_{\alpha,-\theta},$$

$$D_i := x^{-\theta+\epsilon_i}\partial_i, \quad i = 1, \ldots, n,$$

where $\theta = (1, \ldots, 1)$ and $\epsilon_i = (0, \ldots, \underset{i}{1}, \ldots, 0)$. In section 9 we establish that all derivations of (U, Jac_{n+1}^W) are interior. In these sections we also prove that the algebras of interior derivations are isomorphic to Cartan Lie algebras of types W and S, namely, $Int\,(U, Jac_{n+1}^W) \cong W_n(U)$ and $Int\,(U, Jac_n^S) \cong S_n(U)$. These isomorphisms explain index notations on Jac_{n+1}^W and Jac_n^S.

In terms of the n-Lie cohomology [**1**], [**10**], in sections 8 and 9 we calculate the first cohomology groups of the Jacobian algebras with coefficients in adjoint module. Derivations of Lie algebras appear in a natural way in considering central extensions of Lie algebras. Derivations of Lie algebras H_1 and W_n in this sense were described in [**4**]. Our results in the case $n = 2$ are compatible with the results of this paper.

2. Ω-words

Let $\mathbf{Z}$ be the ring of integers, $\mathbf{Z}_+ = \{i \in \mathbf{Z} : i \leq 0\}$ and $\mathbf{Z}^+ = \{i \in \mathbf{Z} : i > 0\}$. Let us given some alphabet $\aleph$ with a map $\aleph \to \mathbf{Z}_+, \alpha \mapsto |\alpha|$, called *arity* map. Let

$$\mathfrak{X} = \{\alpha \in \aleph : |\alpha| = 0\},$$

$$\Omega = \{\omega \in \aleph : |\alpha| > 0\}.$$

Thus, $\aleph = \Omega \cup \mathfrak{X}$. Denote elements of Ω by $\omega_1, \omega_2, \ldots$ and elements of $\mathfrak{X}$ by $x_1, x_2, \ldots$.

Define a *weight map*

$$\aleph \to \mathbf{Z}, \quad \alpha \mapsto ||\alpha||,$$

by

$$||\alpha|| = 1 - |\alpha|.$$

LEMMA 2.1. *The following conditions are equivalent*

- $||\alpha|| \geq 1$
- $||\alpha|| = 1$
- $\alpha \in \mathfrak{X}$
- $|\alpha| = 0$

PROOF. Evident. □

Let $\Gamma(\aleph) = \{a = \alpha_1\alpha_2\cdots\alpha_k\}$ be the set of sequences of elements of $\aleph$. If $a = \alpha_1 \ldots \alpha_k, b = \beta_1 \ldots \beta_s \in \Gamma(\aleph)$, then, by definition, $a = b$ if and only if $k = s$ and $\alpha_1 = \beta_1, \ldots, \alpha_k = \beta_k$. Call the number of elements k of the sequence $\alpha = \alpha_1 \cdots \alpha_k \in \Gamma(\aleph)$ a *length* of α and denote it by $l(\alpha)$. Prolong the weight map $||\ ||$ to

$$||\ || : \Gamma(\aleph) \to \mathbf{Z},$$

by

$$||\alpha_1 \cdots \alpha_k|| = ||\alpha_1|| + \cdots + ||\alpha_k||.$$

Let $\mathbf{Z}^\infty = \{(i_1, i_2, \ldots) : i_1, i_2, \ldots \in \mathbf{Z}\}$. Define a map $\mu : \Gamma(\aleph) \to \mathbf{Z}^\infty$ by

$$\mu(a) = (\mu_k(a), \mu_{k-1}(a), \ldots, \mu_1(a)),$$

$$\mu_i(a) = ||\alpha_i|| + \cdots + ||\alpha_k||, \quad i = 1, 2, \ldots, k,$$

if $a = \alpha_1 \ldots \alpha_k, \alpha_i \in \aleph, i = 1, 2, \ldots, k$.

DEFINITION 2.1. Let $\Gamma_1(\aleph)$ be the subset of $\Gamma(\aleph)$, that consists of elements $a \in \Gamma(\aleph)$, such that

- $||a|| = 1$
- if $a = \alpha_1 \cdots \alpha_k$, $\alpha_i \in \aleph, i = 1, 2, \ldots, k$, then $\mu_i(a) \geq 1$, for any $i = 1, 2, \ldots, k$.

EXAMPLE 2.1. Let $\Omega = \{\omega_3, \omega_2, \omega_2' : |\omega_3| = 3, |\omega_2| = |\omega_2'| = 2\}$. Let $a = \omega_3\omega_2x_1x_2x_3\omega_2'x_4x_5 \in \Gamma(\aleph)$, $b = \omega_3\omega_2x_1x_2\omega_3x_3x_4x_5 \in \Gamma(\aleph)$. Then we have $\mu(a) = (1, 2, 1, 2, 3, 4, 3, 1)$ and $\mu(b) = (1, 2, 3, 1, 2, 3, 2, 0)$. Therefore, $a \in \Gamma_1(\aleph)$ and $b \notin \Gamma_1(\aleph)$.

LEMMA 2.2. *Elements of* $\Gamma_1(\aleph)$ *have the following properties*

- $||\alpha|| = 1, \alpha \in \Gamma_1(\aleph) \Rightarrow \alpha \in \mathfrak{X}$,
- $\Omega \not\subset \Gamma_1(\aleph)$,
- *any element of* $\Gamma_1(\aleph)$ *with length more than 1 begins with some element of* Ω.
- *any element of* $\Gamma_1(\aleph)$ *ends by some element of* $\mathfrak{X}$.

PROOF. Let $a = \alpha_1 \ldots \alpha_k$ be the element of $\Gamma_1(\aleph)$ with length $k = l(a)$.
By definition,

$$\mu_k(a) = ||\alpha_k|| \geq 1$$

Thus, by lemma 2.1, $\alpha_k \in \mathfrak{X}$. So, we have proved that any element of $\Gamma_1(\aleph)$ ends by element of $\mathfrak{X}$. In particular, any element of $\Gamma_1(\aleph)$ with length 1 is an element of $\mathfrak{X}$.

Suppose that $l(a) = k > 1$. By definition,

$$\mu_1(a) = ||\alpha_1 \dots \alpha_k|| = 1, \quad ||\alpha_2 \dots \alpha_k|| \geq 1.$$

Therefore, $||\alpha_1|| \leq 0$. In other words, $\alpha_1 \in \Omega$. So, we established that any element of $\Gamma_1(\aleph)$ with length > 1 begins with element of Ω. □

Define a set of Ω-words [7].

DEFINITION 2.2. (i) Any element of $\mathfrak{X}$ is an Ω-word.
(ii) If $a_1, \dots, a_k$ are Ω-words, then $\omega a_1 \dots a_k$, where $|\omega| = k$, is also a Ω-word.
(iii) Any Ω-word is obtained by these two rules.

Let $a = \omega a_1 \dots a_k$ be some word and α is a word or element of Ω. We say that α enter to a or that α is a part of the word a and write $\alpha \in a$, if one of the following cases are fulfilled,

- α is a word and $\alpha = a$,
- α is a word and α is a part of a_s for some $s = 1, \dots, k$,
- $\alpha \in \Omega$ and $\alpha = \omega$,
- $\alpha \in \Omega$ and α is a part of a_s for some $s = 1, \dots, k$.

For the word $a = \omega\, a_1, \dots, a_k$ define $\omega deg\, a$ or Ω-*degree* of a, by

- $\omega deg\, a = \sum_{j=1}^{k} \omega deg\, a_j + 1$,
- $\omega deg\, x = 0, \quad x \in \mathfrak{X}$,
- $\omega deg\, \omega = 1, \quad \omega \in \Omega$.

So, Ω-degree of a is the number of elements of Ω that enter to a :

$$\omega deg\, a = |\{\omega \in \Omega \cap a\}|.$$

Let $a = \omega a_1 \dots a_k$ be some word. Call $xdeg\, a$ or $\mathfrak{X}$-*degree* of a the number of elements of $\mathfrak{X}$ that enter to a :

- $xdeg\, a = \sum_{j=1}^{k} xdeg\, a_j$,
- $xdeg\, y = 1, \quad y \in \mathfrak{X}$,
- $xdeg\, \omega = 0, \quad \omega \in \Omega$.

A *degree* of a is defined by $deg\, a = xdeg\, a + \omega deg\, a$.

THEOREM 2.1. *The set of* Ω*-words coincides with* $\Gamma_1(\aleph)$.

PROOF. Denote by $\bar{\Gamma}_1(\aleph)$ the set of Ω-words.

Prove that $\bar{\Gamma}_1)(\aleph) \subseteq \Gamma_1(\aleph)$. Let $a \in \bar{\Gamma}_1(\aleph)$. We use induction on $l(a)$. If $l(a) = 1$, then $a = x \in \mathfrak{X}$. Thus, $||x|| = 1$, and $x \in \Gamma_1(\aleph)$. Suppose that any element of $\bar{\Gamma}_1(\aleph)$ with length $< l(a)$ belongs to $\Gamma_1(\aleph)$. If $a = \omega a_1 \dots a_k$, then $l(a_1) < l(a), \dots, l(a_k) < l(a)$. Then by inductive suggestion $a_1, \dots, a_k \in \Gamma_1(\aleph)$. In other words, if $a_i = \alpha_{i,1} \dots \alpha_{i,s_i}$, where $\alpha_{i,j} \in \aleph$, then $\sum_{j=1}^{s_i} ||\alpha_{i,j}|| = 1$. Thus,

$$a = \omega \alpha_{1,1} \dots \alpha_{1,s_1} \dots \alpha_{k,1} \dots \alpha_{k,s_k},$$

and

$$\mu_1(a) = ||a|| = ||\omega|| + \sum_{i=1}^{k} \sum_{j=1}^{s_i} ||\alpha_{i,j}|| = 1 - k + k = 1.$$

By lemma 2.2, $\alpha_{i,s_i} \in \mathfrak{X}$, for any $i = 1, \dots, k$. Therefore, $\mu_i(a) \geq 1$, for any $i \leq \sum_{j=1}^{k} s_j$. So, $a \in \Gamma_1(\aleph)$.

Prove now $\Gamma_1)(\aleph) \subseteq \bar{\Gamma}_1(\aleph)$. By induction on $l(a)$ prove that any $a \in \Gamma_1(\aleph)$ can be presented in the form $a = x \in \mathfrak{X}$, if $l(a) = 1$, or $a = \omega a_1 \dots a_r$, where $||\omega|| = 1 - r, a_1, \dots, a_r \in \Gamma_1(\aleph)$, if $l(a) > 1$.

If $l(a) = 1$, then the statement is trivial. Suppose that our statement is true for elements of $\Gamma_1(\aleph)$ with length $< k$ and $a = \alpha_1 \dots \alpha_k, \alpha_i \in \aleph, i = 1, 2, \dots, k$. Let $\lambda_i = \mu_i(a)$.

Suppose that $\mu_k = 1 \leq \mu_{k-1} \leq \dots \leq \mu_{l+1}$, but $\mu_l < \mu_{l+1}$. This means that $\alpha_k, \dots, \alpha_{l+1} \in \mathfrak{X}$ and $\alpha_l \in \Omega$. Let $|\alpha_l| = q > 0$. Then $q \leq k - l$, since

$$\mu_l = 1 - q + \underbrace{1 + \dots + 1}_{k-l} \geq 1.$$

So, we can consider the element $c = \alpha_l \alpha_{l+1} \dots \alpha_{l+q} \in \Gamma(\aleph)$. The word c is a subword of a. Moreover,

$$||c|| = ||\alpha_l|| + \dots + ||\alpha_{l+q}|| = 1 - q + \underbrace{1 + \dots + 1}_{q} = 1$$

and

$$||\alpha_s \alpha_{s+1} \dots \alpha_{l+q}|| \geq 1,$$

for any $s = l + q, l + q - 1, \dots, l$. So, $c \in \Gamma_1(\aleph)$. By inductive suggestion c is an Ω-word. Therefore, the word

$$b = \beta_1 \dots \beta_{k-q-1} \in \Gamma(\aleph),$$

where

$$\beta_1 = \alpha_1, \dots, \beta_{l-1} = \alpha_{l-1}, \beta_{l+1} = \alpha_{l+q+1}, \dots, \beta_{k-q-1} = \alpha_k,$$

and $\beta_l \in \mathfrak{X}$, has the following properties:

$$l(b) = (l - 1) + 1 + (k - l - q) = k - q - 1 < k,$$

$$||b|| = ||\alpha_1|| + \dots + ||\alpha_{l-1}|| + ||\beta_l|| + ||\alpha_{l+q+1}|| + \dots + ||\alpha_k|| = ||a||,$$

and

$$||\beta_s \dots \beta_{k-q-1}|| \geq 1,$$

for any $s = k - q - 1, k - q - 2, \dots, 1$. These mean that $b \in \Gamma_1(\aleph)$ and $l(b) < l(a)$. By inductive suggestion, $b \in \bar{\Gamma}_1(\aleph)$, and

$$b = \omega b_1 \dots b_r$$

for some $\omega \in \Omega$ and $b_1, \dots, b_r \in \bar{\Gamma}_1(\aleph)$. Since $l(b_1), \dots, l(b_r) < k$, by inductive suggestion $b_1, \dots, b_r \in \Gamma_1(\aleph)$. One of b_s, where $1 \leq s \leq r$, contains β_l. Instead of β_l we can take the Ω-word c and obtain the Ω-word that is equal to a. So, we established that $a \in \bar{\Gamma}_1(\aleph)$.

Our theorem is proved. □

COROLLARY 2.2. *For any Ω-word a,*

$$\sum_{\omega \in \Omega \cap a} |\omega| = \omega deg\, a + x deg\, a - 1.$$

PROOF. If $l(a) = 1$, then $a \in \mathfrak{X}$, and $\omega deg\, a = 0$, $x deg\, a = 1$. So, $\sum_{\omega \in \Omega \cap a} |\omega| = 0 = \omega deg\, a + x deg\, a - 1$.

Suppose that for a, with $l(a) < k$, the statement is true. Let $l(a) = k > 1$. By theorem 2.1, any Ω-word a with length $k > 1$ can be presented in the form

$a = \eta a_1 \dots a_r$, for some Ω-words $a_1, \dots, a_r$ with length $< k$ and some $\eta \in \Omega$ with $|\eta| = r$. Then

$$\Omega \cap a = \{\eta\} \cup \cup_{i=1}^{r} \{\Omega \cap a_i\}.$$

Thus,

$$\sum_{i=1}^{r} \omega deg\, a_i = \omega deg\, a - 1.$$

By inductive suggestion,

$$\sum_{\omega \in \Omega \cap a_i} |\omega| = \omega deg\, a_i + x deg\, a_i - 1,$$

for $i = 1, \dots, r$. Therefore,

$$\sum_{\omega \in \Omega \cap a} |\omega| =$$

$$|\eta| + \sum_{i=1}^{r} \sum_{\omega \in \Omega \cap a_i} |\omega| =$$

$$r + \sum_{i=1}^{r} (\omega deg\, a_i + x deg\, a_i - 1) =$$

$$r + \sum_{i=1}^{r} \omega deg\, a_i + \sum_{i=1}^{r} x deg\, a_i - r =$$

$$\omega deg\, a + x deg\, a - 1.$$

□

COROLLARY 2.3. *Let $\omega deg_i\, a$ be the number of entries of $\omega \in \Omega$ with $|\omega| = i$ in a. Then for any Ω-word a,*

$$x deg\, a = \sum_{i \geq 1} (i-1) \omega deg_i\, a + 1.$$

PROOF. This is another formulation of corollary 2.2. □

COROLLARY 2.4. *Assume that Ω consists of one element ω with $|\omega| = k$. Then for any Ω-word a,*

$$x deg\, a = 1 + (k-1) \omega deg\, a.$$

PROOF. It follows from corollary 2.3 and from the following facts: $\omega deg_k\, a = \omega deg\, a$, and $\omega deg_i\, a = 0$, if $i \neq k$. □

3. Ω-polynomials, Ω-algebras and Ω-identities

DEFINITION 3.1. A linear combination of Ω-words is called *Ω-polynomial*. Polynomial of the form $\lambda_a a$, where a is a word and $\lambda_a \in K$, is called *a monomial*. The monomial is called nontrivial, if $\lambda_a \neq 0$. We say that $\lambda_a a$ is a part of f or $\lambda_a a$ is monomial of f, if $\lambda_a \neq 0$. A space of Ω-polynomials is denoted by $K[\Omega, \mathfrak{X}]$.

Let U and M be some vector spaces. Denote by $T^k(U,M)$ the space of polylinear maps $\psi : \underbrace{U \times \cdots \times U}_{k} \to M$, if $k > 0$, $T^0(U,M) = M$, and $T^k(U,M) = 0$, if $k < 0$. Let $T^*(U,M) = \oplus_k T^k(U,M)$. If $\psi \in T^k(U,M)$, we will write $|\psi| = k$.

Let $\wedge^k(U,M)$ be the subspace of $T^k(U,M)$ consisting of skew-symmetric maps, $\wedge^0(U,M) = M$, $\wedge^k(U,M) = 0$, if $k < 0$ and $\wedge^*(U,M) = \oplus_k \wedge^k (U,M)$.

Let $\Omega = \{\omega_1, \omega_2, \ldots\}$ be some alphabet with an arity map $|\ ,\ | : \Omega \to \mathbf{Z}^+$. Suppose that to each $\omega \in \Omega$ one corresponds some homogeneous map $\omega_U \in T^{|\omega|}(U,U)$. In this case we will say that U has a structure of Ω-*algebra.*

Notice that for any Ω-algebra U and for any Ω-word a one can make substitutions for its parameters $x_i \mapsto u_i \in U$ and $\omega_i \mapsto \omega_{iU}$. The easy way to see it is by presenting a in the form $a = \omega a_1 \ldots a_k$, where $|\omega| = k$ and $a_1, \ldots, a_k$ are words of smaller degree than the *deg a*. One can assume that in $a_1, \ldots, a_k$ our substitutions are correctly defined. Then a would be correctly defined also.

So, for any polynomial $f \in K[\Omega, \mathfrak{X}]$ we can make substitutions in its parameters by elements of U and operations on U. If f depends, say, from parameters $x_1, \ldots, x_k, \omega_1, \ldots, \omega_l$, then we obtain a correctly defined element $f_U = f(u_1, \ldots, u_k, \omega_{1U}, \ldots, \omega_{lU}) \in U$ for any $u_1, \ldots, u_k \in U$.

DEFINITION 3.2. The polynomial $f \in K[\Omega, \mathfrak{X}]$ is called an Ω-*polynomial identity,* or simply, Ω-identity on Ω-algebra U, if

$$f_U(u_1, \ldots, u_k, \omega_{1U}, \ldots, \omega_{lU}) = 0,$$

for any $u_1, \ldots, u_k \in U$.

Let us given two Ω-polynomial identities $f = 0$ and $g = 0$. We say that the identity $f = 0$ *follows* from the identity $g = 0$, and denote $g = 0 \Rightarrow f = 0$, if $f_U = 0$ for any Ω-algebra U, such that $g_U = 0$. The identities $f = 0$ and $g = 0$ are called Ω-*equivalent,* or just equivalent, if $f = 0 \Rightarrow g = 0$ and $g = 0 \Rightarrow f = 0$.

Further, to simplify the notation we will often identify the polynomial $f = f(t_1, \ldots, t_k, \omega_1, \ldots, \omega_l)$ with $f_U = f(u_1, \ldots, u_k, \omega_{1U}, \ldots, \omega_{lU}) \in U$ (the result of substitution) and call $f(u_1, \ldots, u_k, \omega_{1U}, \ldots, \omega_{lU})$ shortly a Ω-polynomial, or just a polynomial. Notice that a formal definition of Ω-words does not need any brackets and comma's, but for practical use it is more convenient to use brackets and comma's. We will use brackets keeping in mind that we will do it from the right to the left as in the proof of theorem 2.1.

EXAMPLE 3.1. Let $\Omega = \{\omega_3, \omega_2, \omega_2' : |\omega_3| = 3, |\omega_2| = |\omega_2'| = 2$. Let $a = \omega_3\omega_2 x_1 x_2 x_3 \omega_2' x_4 x_5 \in \Gamma_1(\aleph)$, Then for any Ω-algebra U and for any $u_1, \ldots, u_5 \in U$,

$$a_U = \omega_{3U}(\omega_{2U}(u_1, u_2), u_3, \omega_{2U}'(u_4, u_5)) \in U,$$

or simply,

$$a = \omega_3(\omega_2(x_1, x_2), x_3, \omega_2'(x_4, x_5)).$$

4. 3-Lie algebras

THEOREM 4.1. *If $p = 0$ or $p > 3$, then any 3-Lie-Poisson algebra is strong.*

PROOF. Let $(U, \cdot, \omega)$ be 3-Lie-Poisson. Recall that

$$r^\omega(u_1, \ldots, u_6) =$$

$$\omega(u_1, u_2, u_3) \cdot \omega(u_4, u_5, u_6 - \omega(u_1, u_2, u_5) \cdot \omega(u_4, u_3, u_6)$$

$$+\omega(u_1,u_2,u_6)\cdot\omega(u_4,u_3,u_5)+\omega(u_4,u_2,u_3)\cdot\omega(u_1,u_5,u_6)$$
$$-\omega(u_4,u_2,u_5)\cdot\omega(u_1,u_3,u_6)+\omega(u_4,u_2,u_6)\cdot\omega(u_1,u_3,u_5)$$

is symmetric in two arguments u_1 and u_4 and skew-symmetric in three arguments u_3,u_5,u_6.

By theorem 6 [**2**], for 3-Lie-Poisson algebras, $r^\omega(u_1,\dots,u_6)=0$, for any $u_1,\dots,u_6\in U$. One can check that

$$3f^\omega(u_1,u_2,u_3,u_4,u_5,u_6)=$$
$$2r^\omega(u_1,u_2,u_3,u_4,u_5,u_6)+r^\omega(u_2,u_3,u_1,u_4,u_5,u_6)$$
$$-r^\omega(u_2,u_4,u_1,u_3,u_5,u_6)+r^\omega(u_2,u_5,u_1,u_3,u_4,u_6)-r^\omega(u_2,u_6,u_1,u_3,u_4,u_5).$$

So, $f^\omega=0$ is also identity for $(U,\cdot,\omega)$, if $p\neq 3$. □

PROPOSITION 4.1. $f^\omega=0\Rightarrow r^\omega=0$.

PROOF. One can check that:

$$-r^\omega(u_1,u_n,u_2,\dots,\hat{u_n},\dots,u_{2n})=$$
$$f^\omega(u_1,u_2,\dots,u_n,\dots,u_{2n})+(-1)^n f^\omega(u_2,\dots,u_n,u_1,u_n,\dots,u_{2n}).$$

Therefore, the identity $r^\omega=0$ follows from the identity $f^\omega=0$. □

REMARK. Notice that for $n=2$ the identities $r^\omega=0$ and $f^\omega=0$ are different. More exactly, the identity $r^\omega=0$ does not appear, if $n=2$. There exist 2-Lie-Poisson algebras that are not strong. Let us give an example of such algebras. Let $K_{2l}=K[x_1,\dots,x_{2l}]$ and $\omega=\sum_{i=1}^{l}\partial_i\wedge\partial_{i+l}$. Then $(K_{2l},\cdot,\omega)$ is 2-Lie-Poisson. It is easy to check that $(K_2,\cdot,\partial_1\wedge\partial_2)$ satisfies the identity $f^{\partial_1\wedge\partial_2}=0$. If $l>1$, the polynomial

$$f^\omega(a,u,v,w)=\omega(a,u)\cdot\omega(v,w)+\omega(a,v)\cdot\omega(w,u)+\omega(a,w)\cdot\omega(u,v)$$

is not an identity. For instance,

$$f^\omega(x_1,x_2,x_3,x_4)=\omega(x_1,x_2)\cdot\omega(x_3,x_4)=1\neq 0.$$

So, the algebra $(K_{2l},\cdot,\omega)$ is strong 2-Lie-Poisson, if and only if $l=1$.

5. Minimal identities for 3-Jacobians

THEOREM 5.1. $(p\neq 2,3)$ *Any polynomial identity of* Ω*-degree* 2 *of Jacobian algebra* $(K[x_1,x_2,x_3],Jac_3^S)$ *follows from* 3*-Lie and skew-symmetric identities for* 3*-multiplication* Jac_3^S.

PROOF. Let $\omega=Jac_3^S$. Define polynomials g^ω, h^ω and q^ω by

$$g^\omega=g^\omega(t_1,\dots,t_5)=$$
$$\omega t_1t_2\omega t_3t_4t_5-\omega\omega t_1,t_2,t_3t_4t_5+$$
$$\omega\omega t_1t_2t_4t_3t_5-\omega\omega t_1t_2t_5t_3t_4,$$

$$h^\omega=h^\omega(t_1,\dots,t_5)=$$
$$\omega t_1t_2\omega t_3t_4t_5-\omega t_1t_3\omega t_2t_4t_5+$$
$$\omega t_1t_4\omega t_2t_3t_5-\omega t_1t_5\omega t_2t_3t_4,$$

$$q^\omega=q^\omega(t_1,\dots,t_5)=$$

$$\sum_{i<j<5} (-)^{i+j}\omega t_i t_j \omega t_1 \dots \hat{t_i} \dots \hat{t_j} \dots t_5.$$

Notice that

$$3h^\omega(t_1,t_2,t_3,t_4,t_5) = \sum_{2\le i<j\le 5} (-1)^{i+j} g^\omega(t_i,t_j,t_1,\dots,\hat{t_i},\dots\hat{t_j},\dots,t_5).$$

Further,

$$\sum_{1\le i<j\le 5} (-1)^{i+j} g^\omega(t_1,\dots,t_5) = 2\sum_{1\le i<j\le 5} (-1)^{i+j}\omega t_i t_j \omega t_1 \dots \hat{t_i} \dots \hat{t_j} \dots t_5.$$

Therefore,

$$2q^\omega(t_2,t_3,t_4,t_5,t_1) = \sum_{1\le i<j\le 5} (-1)^{i+j} g^\omega(t_1,\dots,t_5) - 2h^\omega(t_1,\dots,t_5).$$

So, $h^\omega = 0$ and $q^\omega = 0$ are identities on (U,ω), if $g^\omega = 0$ is an identity and $p \neq 2,3$.

Let f be any polynomial of $wdeg\, f = 2$. Since $|\omega| = 3$, by corollary 2.3 we should prove that any polynomial of the form

$$f(t_1,\dots,t_5) = \sum_{i_1<i_2,i_3<i_4<i_5} \lambda_{i_1 i_2}\omega(t_{i_1},t_{i_2},\omega(t_{i_3},t_{i_4},t_{i_5})),$$

such that

$$f(u_1,\dots,u_5) = 0,$$

for any $u_1,\dots,u_5 \in U = K[x_1,x_2,x_3]$, is a linear combination of polynomials that can be obtained from the polynomial g^ω by permutation of variables $t_1,\dots,t_5$.

We have

$$f(x_1,x_2,x_1,x_2,x_3^2) = 0 \Rightarrow \lambda_{12} - \lambda_{14} + \lambda_{34} - \lambda_{23} = 0,$$
$$f(x_2,x_1,x_1,x_2,x_3^2) = 0 \Rightarrow \lambda_{12} + \lambda_{13} + \lambda_{24} + \lambda_{34} = 0,$$
$$f(x_1,x_2,x_1,x_3^2,x_2) = 0 \Rightarrow \lambda_{12} + \lambda_{15} - \lambda_{23} - \lambda_{35} = 0,$$
$$f(x_2,x_1,x_1,x_3^2,x_2) = 0 \Rightarrow \lambda_{12} + \lambda_{13} - \lambda_{25} - \lambda_{35} = 0,$$
$$f(x_1,x_2,x_3^2,x_1,x_2) = 0 \Rightarrow \lambda_{12} + \lambda_{15} + \lambda_{24} + \lambda_{45} = 0,$$
$$f(x_2,x_1,x_3^2,x_1,x_2) = 0 \Rightarrow -\lambda_{12} + \lambda_{14} + \lambda_{25} - \lambda_{45} = 0.$$

The obtained system of linear equations has rank 5 and the parameters λ_{15}, λ_{25}, λ_{34}, λ_{35}, λ_{45} can be chosen as free. So, any polynomial f, such that $xdeg\, f = 5$ and $f = 0$ is an identity on $(K[x_1,x_2,x_3],\omega)$, is a linear combination of the following five polynomials

$$f_{45} = -\omega t_1 t_2 \omega t_3 t_4 t_5 + \omega t_1 t_3 \omega t_2 t_4 t_5 - \omega t_2 t_3 \omega t_1 t_4 t_5 + \omega t_4 t_5 \omega t_1 t_2 t_3,$$
$$f_{35} = \omega t_1 t_2 \omega t_3 t_4 t_5 + \omega t_1 t_4 \omega t_2 t_3 t_5 - \omega t_2 t_4 \omega t_1 t_3 t_5 + \omega t_3 t_5 \omega t_1 t_2 t_4,$$
$$f_{34} = \omega t_1 t_2 \omega t_3 t_4 t_5 - \omega t_1 t_3 \omega t_2 t_4 t_5 + \omega t_1 t_4 \omega t_2 t_3 t_5 + \omega t_2 t_3 \omega t_1 t_4 t_5 - \omega t_2 t_4 \omega t_1 t_3 t_5 + \omega t_3 t_4 \omega t_1 t_2 t_5,$$
$$f_{25} = \omega t_1 t_2 \omega t_3 t_4 t_5 + \omega t_2 t_3 \omega t_1 t_4 t_5 - \omega t_2 t_4 \omega t_1 t_3 t_5 + \omega t_2 t_5 \omega t_1 t_3 t_4,$$
$$f_{15} = -\omega t_1 t_2 \omega t_3 t_4 t_5 + \omega t_1 t_3 \omega t_2 t_4 t_5 - \omega t_1 t_4 \omega t_2 t_3 t_5 + \omega t_1 t_5 \omega t_2 t_3 t_4.$$

We see that

$$f_{45} = g^\omega(t_4,t_5,t_1,t_2,t_3),$$

$$f_{35} = g^{\omega}(t_3, t_5, t_1, t_2, t_4),$$
$$f_{34} = -q^{\omega},$$
$$f_{25} = -h^{\omega}(t_2, t_1, t_3, t_4, t_5),$$
$$f_{15} = -h^{\omega}.$$

So, any identity of $\mathfrak{X}$-degree 5 follows from 3-Lie identity $g^{\omega} = 0$ and skew-symmetric identity. □

CONJECTURE. Let $p = 0$ and $n > 2$. Any identity of Ω-degree ≤ 3 of n-Lie algebra $(K[x_1, \ldots, x_n], Jac_n^S)$ follows from n-Lie and skew-symmetric identities. Any identity of Ω-degree ≤ 3 of n-Lie-Poisson algebra $(K[x_1, \ldots, x_n], \cdot, Jac_n^S)$ follows from n-Lie and skew-symmetric identities for the Jacobian, the Leibniz rule for $\cdot$ and the identity $f^{Jac_n^S} = 0$.

Here we suppose that commutativity and associativity identities for a binary operation are given. Definitions of n-Lie-Poisson algebras and the polynomial f^{ω} are given in the next section.

6. Constructions of n-Lie algebras from n-Lie-Poisson algebras

Usually n-Lie algebras are considered for $n > 1$. Complete this definition by considering the case $n = 1$. Call any vector space U with a linear map $f : U \to U$, i.e., (U, f) a 1-Lie algebra.

DEFINITION 6.1. Let $A = (U, \cdot, \omega)$ be an algebra with two operations: $(U, U) \to U, (u, v) \mapsto u \cdot v$ be a bilinear multiplication and $\omega : \wedge^n U \to U$ be a skew-symmetric n-linear multiplication. We say that A is *n-Lie-Poisson,* if

- $(U, \cdot)$ is an associative commutative algebra
- (U, ω) is an n-Lie algebra.
- $\omega(u \cdot u', u_2, \ldots, u_n) = \omega(u, u_2, \ldots, u_n) \cdot u' + u \cdot \omega(u', u_2, \ldots, u_n)$, for any $u, u', u_2, \ldots, u_n \in U$.

Consider on U one more polynomial

$$f^{\omega}(u_1, \ldots, u_{n-1}, v_1, \ldots, v_n) =$$
$$\sum_{i=1}^{n+1} (-1)^i \omega(u_1, \ldots, u_{n-1}, v_i) \cdot \omega(v_1, \ldots, \hat{v_i}, \ldots, v_{n+1}).$$

Let us given an algebra $(U, \cdot, \omega)$ with one binary operation $(u, v \mapsto u \cdot v$ and one n-ary operation $\wedge^n U \to U, (u_1, \ldots, u_n) \mapsto \omega(u_1, \ldots, u_n)$.

DEFINITION 6.2. An n-Lie-Poisson algebra is called *strong*, if it satisfies the identity $f^{\omega} = 0$.

EXAMPLE 6.1. We will see below that the Jacobian algebra $(U, \cdot, Jac_n^S)$ is strong n-Lie-Poisson and that the algebra $(U, \cdot, Jac_{n+1}^W)$ satisfies the identity $f^{\omega} = 0$.

Let $(U, \cdot, \omega)$ be n-Lie-Poisson algebra. Recall that $D \in Der\,(U, \cdot, \omega)$, if

- $D \in Der\,(U, \cdot)$, i.e., $D(u \cdot v) = D(u) \cdot v + u \cdot D(v)$,
- $D \in Der\,(U, \omega)$, i.e.,

$$D(\omega(u_1, \ldots, u_n)) = \sum_{i=1}^{n} \omega(u_1, \ldots, u_{i-1}, Du_i, u_{i+1}, \ldots, u_n).$$

Call in a such case D a n-Lie-Poisson derivation.

EXAMPLE 6.2. Let $n = 1$. Then $(U, \cdot, \omega)$ is n-Lie-Poisson, if $\omega : U \to U$ is a derivation of $(U, \cdot)$. If D is an n-Lie-Poisson derivation of $(U, \cdot, \omega)$, then $[D, \omega] = 0$.

THEOREM 6.1. *Let $(U, \cdot, \omega)$ be a strong n-Lie-Poisson algebra. Consider $D \in Der\,(U, \cdot, \omega)$. Construct on U a new skew-symmetric $(n+1)$-multiplication $\bar{\omega} = D \wedge \omega$. Then $(U, \cdot, \bar{\omega})$ is strong $(n+1)$-Lie-Poisson.*

PROOF. We have

$$\bar{\omega}(u \cdot u', u_1, \ldots, u_n) =$$

$$D(u \cdot u') \cdot \omega(u_1, \ldots, u_n) + \sum_{i=1}^{n} (-1)^i D(u_i) \cdot \omega(u \cdot u', u_1, \ldots, \hat{u_i}, \ldots, u_n) =$$

$$(D(u) \cdot u') \cdot \omega(u_1, \ldots, u_n) + u \cdot D(u') \cdot \omega(u_1, \ldots, u_n)$$
$$+ \sum_{i=1}^{n} (-1)^i D(u_i) \cdot u \cdot \omega(u', u_1, \ldots, \hat{u_i}, \ldots, u_n) +$$
$$\sum_{i=1}^{n} (-1)^i D(u_i) \cdot \omega(u, u_1, \ldots, \hat{u_i}, \ldots, u_n) \cdot u' =$$

$$(D(u) \cdot \omega(u_1, \ldots, u_n)) \cdot u' + \sum_{i=1}^{n} (-1)^i (D(u_i) \cdot \omega(u, u_1, \ldots, \hat{u_i}, \ldots, u_n)) \cdot u' +$$

$$u \cdot (D(u') \cdot \omega(u_1, \ldots, u_n)) + \sum_{i=1}^{n} (-1)^i u \cdot (D(u_i) \cdot \omega(u', u_1, \ldots, \hat{u_i}, \ldots, u_n)) =$$

$$D \wedge \omega(u, u_1, \ldots, u_n) \cdot u' + u \cdot \omega(u', u_1, \ldots, u_n)$$

Further,

$$\sum_{i=1}^{n+2} (-1)^i D \wedge \omega(u_1, \ldots, u_n, v_i) \cdot D \wedge \omega(v_1, \ldots, \hat{v_i}, \ldots, v_{n+2}) =$$

$$X_1 + X_2,$$

where

$$X_1 = \sum_{i=1}^{n+2} \sum_{j=1}^{n} \sum_{1 \le s \le n+2, s \ne i} (-1)^{i+j+s+\delta(s>i)} \cdot$$
$$D(u_j) \cdot D(v_s) \cdot \omega(u_1, \ldots, \hat{u_j}, \ldots, u_n, v_i) \cdot \omega(v_1, \ldots, \hat{v_s}, \ldots, \hat{v_i}, \ldots, v_{n+2}),$$

$$X_2 = \sum_{i=1}^{n+2} \sum_{1 \le s \le n+2, s \ne i} (-1)^{i+n+s+\delta(s>i)+1} \cdot$$
$$D(v_i) \cdot D(v_s) \cdot \omega(u_1, \ldots, u_n) \cdot \omega(v_1, \ldots, \hat{v_s}, \ldots, \hat{v_i}, \ldots, v_{n+2}).$$

Notice that, by identity $f^{\omega} = 0$,

$$X_1 = \sum_{j=1}^{n} \sum_{1 \le s \le n+2} (-1)^{j+s} D(u_j) \cdot D(v_s) \cdot$$

$$\left(\sum_{1 \le i \le n+2, i \ne s} (-1)^{i+\delta(s>i)} \omega(u_1, \ldots, \hat{u_j}, \ldots, u_n, v_i) \cdot \omega(v_1, \ldots, \hat{v_s}, \ldots, \hat{v_i}, \ldots, v_{n+2}) \right)$$

$$= 0$$

and by commutativity of multiplication $\cdot$,

$$X_2 = (-1)^n \omega(u_1, \dots, u_n) \cdot$$

$$\left(\sum_{i=1}^{n+2} \sum_{i<s\le n+2} (-1)^{i+s} D(v_i) \cdot D(v_s) \cdot \omega(v_1, \dots, \hat{v_i}, \dots, \hat{v_s}, \dots, v_{n+2}) - \right.$$

$$\left. \sum_{i=1}^{n+2} \sum_{i<s\le n+2} (-1)^{i+s} D(v_s) \cdot D(v_i) \cdot \omega(v_1, \dots, \hat{v_i}, \dots, \hat{v_s}, \dots, v_{n+2}) \right) = 0$$

So, $f^{\bar{\omega}} = 0$ is the identity.

Notice that

$$D \wedge \omega(u_1, \dots, u_n, D \wedge \omega(u_{n+1}, \dots, u_{2n+1}))$$

$$- \sum_{i=1}^{n+1} (-1)^{i+n+1} D \wedge \omega(D \wedge \omega(u_1, \dots, u_n, u_{n+i}), u_{n+1}, \dots, \widehat{u_{n+i}}, \dots, u_{2n+1}) =$$

$$\sum_{i=1}^{2n+1} D(D(u_i)) \cdot g_i(u_1, \dots, \hat{u_i}, \dots, u_{2n+1}) +$$

$$\sum_{1\le i,j\le 2n+1, i\ne j} D(u_i) \cdot D(u_j) \cdot g_{i,j}(u_1, \dots, \hat{u_i}, \dots, \hat{u_j}, \dots, u_{2n+1}) +$$

$$\sum_{i=1}^{2n+1} \sum_{j=1}^{2n+1} D(u_i) \cdot h_{i,j}(u_1, \dots, \hat{u_i}, \dots, u_{j-1}, D(u_j), u_{j+1}, \dots, u_{2n+1}),$$

for some polynomials $g_i, g_{i,j}$ and $h_{i,j}$, that do not depend from D.

The following relations can be obtained by tedious calculations.

If $n < i \le 2n+1$, then $g_i = 0$. If $1 \le i \le n$, then by the identity $f^\omega = 0$,

$$g_i(u_1, \dots, \hat{u_i}, \dots, u_{2n+1}) =$$

$$(-1)^{i+n+1} \sum_{j=n+1}^{2n+1} (-1)^j \omega(u_1, \dots, \hat{u_i}, \dots, u_n, u_j) \cdot \omega(u_{n+1}, \dots, \hat{u_j}, \dots, u_{2n+1}) =$$

$$(-1)^{i+n+1} f^\omega(u_1, \dots, \hat{u_i}, \dots, u_{2n+1}) = 0.$$

If $1 \le i, j \le n$, or $n < i, j \le 2n+1$, then $g_{i,j} = 0$. If $1 \le i \le n$, $n < j \le 2n+1$, by n-Lie identity $g_{i,j} = 0$.

If $n < j \le 2n+1$, then $h_{i,j} = 0$. If $1 \le j \le n$, then

$$h_{i,j} = \pm f^\omega(u_1, \dots, \hat{u_i}, \dots, u_{j-1}, D(u_j), u_{j+1}, \dots, u_n, u_{n+1}, \dots, u_{2n+1}),$$

in the case of $1 \le i \le n$, and

$$h_{i,j} = \pm f^\omega(u_1, \dots, \hat{u_j}, \dots, u_n, u_{n+1}, \dots, u_{i-1}, D(u_i), u_{i+1}, \dots, u_{2n+1}),$$

in the case of $n < i \le 2n+1$. Thus, in both cases, by the identity $f^\omega = 0$, we have $h_{i,j} = 0$.

So, the multiplication $\bar{\omega}$ satisfies $(n+1)$-Lie identity.

□

COROLLARY 6.2. *Let U be an associative commutative algebra with commuting derivations $\partial_1, \dots, \partial_n$. Let $\omega = \partial_1 \wedge \cdots \wedge \partial_n$ Then $(U, \cdot, \omega)$ is strong n-Lie-Poisson.*

The algebra $(U, \cdot, \omega)$ constructed in corollary 6.2 is called *n-Lie Jacobian algebra of type S*.

PROOF. We will argue by induction on n. If $n = 1$, then $(U, \cdot, \partial_1)$ is 1-Lie-Poisson, if and only if $\partial_1 \in Der\,(U, \cdot)$. In this case the identity $f^\omega = 0$ follows from the commutativity law for the algebra $(U, \cdot)$.

Suppose that the algebra $(U, \cdot, \partial_1 \wedge \cdots \wedge \partial_{n-1})$ is strong $(n-1)$-Lie-Poisson. By definition $\partial_n \in Der\,(U, \cdot)$. Since $[\partial_n, \partial_i] = 0$, for any $i = 1, \ldots, n-1$, then

$$\partial_n(\partial_1 \wedge \cdots \wedge \partial_{n-1}(u_1, \ldots, u_{n-1}) =$$

$$\partial_n(\sum_{\sigma \in Sym_{n-1}} sign\,\sigma\, \partial_{\sigma(1)} u_1 \cdot \cdots \cdot \partial_{\sigma(n-1)} u_{n-1}) =$$

$$\sum_{i=1}^{n-1} \sum_{\sigma \in Sym_{n-1}} sign\,\sigma\, \partial_{\sigma(1)} u_1 \cdots \cdots \partial_{\sigma(i-1)} u_{i-1} \partial_n(\partial_{\sigma(i)} u_i) \cdot \partial_{\sigma(i+1)} u_{i+1} \cdots \cdots \partial_{\sigma(n-1)} u_{n-1} =$$

$$\sum_{i=1}^{n-1} \sum_{\sigma \in Sym_{n-1}} sign\,\sigma\, \partial_{\sigma(1)} u_1 \cdots \cdots \partial_{\sigma(i-1)} u_{i-1} \cdot \partial_{\sigma(i)} \partial_n(u_i) \cdot \partial_{\sigma(i+1)} u_{i+1} \cdots \cdots \partial_{\sigma(n-1)} u_{n-1} =$$

$$\sum_{i=1}^{n-1} \partial_1 \wedge \cdots \wedge \partial_{n-1}(u_1, \ldots, u_{i-1}, \partial_n u_i, u_{i+1}, \ldots, u_{n-1}).$$

So, $\partial_n \in Der\,(U, \partial_1 \wedge \cdots \wedge \partial_{n-1})$. Thus we can apply theorem 6.1. By this theorem we obtain that $(U, \cdot, \partial_1 \wedge \cdots \wedge \partial_n)$ is strong n-Lie-Poisson. □

THEOREM 6.3. *Let $(U, \cdot, \omega)$ be strong n-Lie-Poisson. Endow U with a new skew-symmetric $(n+1)$-multiplication $\tilde{\omega} = id \wedge \omega$, where $id : U \to U, u \mapsto u$, is the identity map. Then $(U, \tilde{\omega})$ is $(n+1)$-Lie algebra. It satisfies the identity $f^{\tilde{\omega}} = 0$. If U has unit 1, then it satisfies one more identity*

$$\tilde{\omega}(u \cdot u', u_1, \ldots, u_n) - u \cdot \tilde{\omega}(u', u_1, \ldots, u_n) - u' \cdot \tilde{\omega}(u, u_1, \ldots, u_n) = -(u \cdot u') \cdot i(1)\tilde{\omega}(u_1, \ldots, u_n) \tag{4}$$

PROOF. Since $L_{u_2, \ldots, u_n} \in Der\,(U, \cdot)$, and $1 \cdot 1 = 1$,

$$i(1)\omega = 0.$$

Therefore,

$$i(1)(id \wedge \omega) = \omega \tag{5}$$

So, for any $u, u', u_1, \ldots, u_n \in U$,

$$\tilde{\omega}(u \cdot u', u_1, \ldots, u_n) = Y_1 + Y_2,$$
$$u \cdot \tilde{\omega}(u', u_1, \ldots, u_n) = Z_1 + Z_2,$$
$$u' \cdot \tilde{\omega}(u, u_1, \ldots, u_n) = W_1 + W_2,$$

where

$$Y_1 = (u \cdot u') \cdot \omega(u_1, \ldots, u_n),$$
$$Y_2 = \sum_{i=1}^{n} (-1)^i u_i \omega(u \cdot u', u_1, \ldots, \hat{u_i}, \ldots, u_n),$$
$$Z_1 = u \cdot (u' \cdot \omega(u_1, \ldots, u_n)),$$
$$Z_2 = \sum_{i=1}^{n} (-)^i u \cdot (u_i \cdot \omega(u', u_1, \ldots, \hat{u_i}, \ldots, u_n)),$$

$$W_1 = u' \cdot (u \cdot \omega(u_1, \dots, u_n)),$$

$$W_2 = \sum_{i=1}^{n} (-)^i u' \cdot (u_i \cdot \omega(u, u_1, \dots, \hat{u_i}, \dots, u_n)).$$

Since $L_{u_1,\dots,\hat{u_i},\dots,u_n} \in Der\,(U, \cdot)$,

$$Y_1 = Z_1 = W_1, \quad Y_2 = Z_2 + W_2.$$

Therefore, by (5), the identity (4) is true.

We have

$$f^{\tilde{\omega}}(u_1, \dots, u_n, v_1, \dots, v_{n+2}) =$$

$$\sum_{i=1}^{n+2} (-1)^i id \wedge \omega(u_1, \dots, u_n, v_i) \cdot \tilde{\omega}(v_1, \dots, \hat{v_i}, \dots, v_{n+2}) =$$

$$\sum_{i=1}^{n+2} \sum_{j=1}^{n} (-1)^{i+j} u_j \cdot \omega(u_1, \dots, \hat{u_j}, \dots, u_n, v_i) \cdot \tilde{\omega}(v_1, \dots, \hat{v_i}, \dots, v_{n+2}) +$$

$$\sum_{i=1}^{n+2} (-1)^{i+n+1} v_i \cdot \omega(u_1, \dots, u_n) \cdot \tilde{\omega}(v_1, \dots, \hat{v_i}, \dots, v_{n+2}) =$$

$$T_1 + T_2,$$

where

$$T_1 = \sum_{i=1}^{n+2} \sum_{j=1}^{n} \sum_{1 \le s \le n+2, s \ne i} (-1)^{i+j+s+\delta(s>i)} \cdot$$

$$u_j \cdot v_s \cdot \omega(u_1, \dots, \hat{u_j}, \dots, u_n, v_i) \cdot \omega(v_1, \dots, \hat{v_s}, \dots, \hat{v_i}, \dots, v_{n+2}),$$

$$T_2 = \sum_{i=1}^{n+2} \sum_{1 \le s \le n+2, s \ne i}$$

$$(-1)^{i+s+\delta(s>i)+n+1} v_i \cdot v_s \cdot \omega(u_1, \dots, u_n) \cdot \omega(v_1, \dots, \hat{v_s}, \dots, \hat{v_i}, \dots, v_{n+2}).$$

Since $(U, \cdot)$ is commutative,

$$T_2 = (-1)^{n+1} \cdot \omega(u_1, \dots, u_n) \cdot$$

$$\left(\sum_{i=1}^{n+2} \sum_{1 \le s \le n+2, s \ne i} (-1)^{i+s+\delta(s>i)} v_i \cdot v_s \cdot \omega(v_1, \dots, \hat{v_s}, \dots, \hat{v_i}, \dots, v_{n+2}) \right) = 0.$$

Notice that

$$T_1 = \sum_{j=1}^{n} \sum_{s=1}^{n+2} (-1)^{j+s} u_j \cdot v_s \cdot$$

$$\left(\sum_{1 \le i \le n+2, i \ne s} (-1)^{i+\delta(s>i)} \omega(u_1, \dots, \hat{u_j}, \dots, u_n, v_i) \cdot \omega(v_1, \dots, \hat{v_s}, \dots, \hat{v_i}, \dots, v_{n+2}) \right) =$$

$$\sum_{j=1}^{n} \sum_{s=1}^{n+2} (-1)^{j+s} u_j \cdot v_s \cdot f^{\omega}(u_1, \dots, \hat{u_j}, \dots, u_n, v_1, \dots, \hat{v_s}, \dots, v_{n+2}).$$

Since $f^{\omega} = 0$ is identity on $(U, \cdot, \omega)$, we have $T_1 = 0$. So, the algebra $(U, \cdot, \tilde{\omega})$ satisfies the identity $f^{\tilde{\omega}} = 0$.

We have

$$id \wedge \omega(u_1, \ldots, u_n, id \wedge \omega(u_{n+1}, \ldots, u_{2n+1})) =$$

$$\sum_{i=n+1}^{2n+1} (-1)^{i+n+1} id \wedge \omega(u_1, \ldots, u_n, u_i \cdot \omega(u_{n+1}, \ldots, \hat{u_i}, \ldots, u_{2n+1})) =$$

$$\sum_{j=1}^{n} \sum_{i=n+1}^{2n+1} (-1)^{i+j+n} u_j \cdot \omega(u_1, \ldots, \hat{u_j}, \ldots, u_n, u_i \cdot \omega(u_n, \ldots, \hat{u_i}, \ldots, u_{2n+1})) +$$

$$\sum_{i=n+1}^{2n+1} (-1)^{i+1} u_i \cdot \omega(u_1, \ldots, u_n) \cdot \omega(u_{n+1}, \ldots, \hat{u_i}, \ldots, u_{2n+1}) =$$

$$A_1 + A_2 + A_3,$$

where

$$A_1 = \sum_{j=1}^{n} \sum_{i=n+1}^{2n+1} (-1)^{i+j+n} u_j \cdot \omega(u_1, \ldots, \hat{u_j}, \ldots, u_n, u_i) \cdot \omega(u_n, \ldots, \hat{u_i}, \ldots, u_{2n+1})),$$

$$A_2 = \sum_{j=1}^{n} \sum_{i=n+1}^{2n+1} (-1)^{i+j+n} u_j u_i \cdot \omega(u_1, \ldots, \hat{u_j}, \ldots, u_n, \omega(u_n, \ldots, \hat{u_i}, \ldots, u_{2n+1})),$$

$$A_3 = \omega(u_1, \ldots, u_n) \cdot \sum_{i=n+1}^{2n+1} (-1)^{i+1} u_i \cdot \omega(u_{n+1}, \ldots, \hat{u_i}, \ldots, u_{2n+1}).$$

On the other hand,

$$\sum_{i=n+1}^{2n+1} (-1)^{i+n+1} (id \wedge \omega)(id \wedge \omega(u_1, \ldots, u_n, u_i), u_{n+1}, \ldots, \hat{u_i}, \ldots, u_{2n+1}) =$$

$$\sum_{i=n+1}^{2n+1} \sum_{j=1}^{n} (-1)^{i+j+n} (id \wedge \omega)(u_j \cdot \omega(u_1, \ldots, \hat{u_j}, \ldots, u_n, u_i), u_{n+1}, \ldots, \hat{u_i}, \ldots, u_{2n+1}) +$$

$$\sum_{i=n+1}^{2n+1} (-1)^{i+1} (id \wedge \omega)(u_i \cdot \omega(u_1, \ldots, u_n), u_{n+1}, \ldots, \hat{u_i}, \ldots, u_{2n+1}) =$$

$$B_1 + B_2 + B_3 + B_4,$$

where

$$B_1 = \sum_{i=n+1}^{2n+1} \sum_{j=1}^{n} (-1)^{i+j+n} u_j \cdot \omega(u_1, \ldots, \hat{u_j}, \ldots, u_n, u_i) \cdot \omega(u_{n+1}, \ldots, \hat{u_i}, \ldots, u_{2n+1}),$$

$$B_2 = \sum_{i=n+1}^{2n+1} \sum_{j=1}^{n} \sum_{n<s\leq 2n+1, s\neq i} (-1)^{i+j+s+1+\delta(s>i)}.$$

$$u_s \cdot \omega(u_j \cdot \omega(u_1, \ldots, \hat{u_j}, \ldots, u_n, u_i), u_{n+1}, \ldots, \hat{u_s}, \ldots, \hat{u_i}, \ldots, u_{2n+1}),$$

$$B_3 = \omega(u_1, \dots, u_n) \cdot \sum_{i=n+1}^{2n+1} (-1)^{i+1} u_i \cdot \omega(u_{n+1}, \dots, \hat{u_i}, \dots, u_{2n+1}),$$

$$B_4 = \sum_{i=n+1}^{2n+1} \sum_{n<s\le 2n+1, s\neq i} (-1)^{i+s+n+\delta(s>i)}.$$

$$u_s \cdot \omega(u_i \cdot \omega(u_1, \dots, u_n), u_{n+1}, \dots, \hat{u_s}, \dots, \hat{u_i}, \dots, u_{2n+1}).$$

Notice that

$$A_1 = B_1,$$

$$A_3 = B_3.$$

So, to check $(n+1)$-Lie identity for $(n+1)$-multiplication $\tilde{\omega}$ we must prove, that

$$A_2 = B_2 + B_4.$$

We have

$$B_2 = B_{2,1} + B_{2,2},$$

$$B_4 = B_{4,1} + B_{4,2},$$

where

$$B_{2,1} = \sum_{i=n+1}^{2n+1} \sum_{j=1}^{n} \sum_{n<s\le 2n+1, s\neq i} (-1)^{i+j+s+1+\delta(s>i)}.$$

$$u_s \cdot u_j \cdot \omega(\omega(u_1, \dots, \hat{u_j}, \dots, u_n, u_i), u_{n+1}, \dots, \hat{u_s}, \dots, \hat{u_i}, \dots, u_{2n+1}),$$

$$B_{2,2} = \sum_{i=n+1}^{2n+1} \sum_{j=1}^{n} \sum_{n<s\le 2n+1, s\neq i} (-1)^{i+j+s+1+\delta(s>i)}.$$

$$u_s \cdot \omega(u_1, \dots, \hat{u_j}, \dots, u_n, u_i) \cdot \omega(u_j, u_{n+1}, \dots, \hat{u_s}, \dots, \hat{u_i}, \dots, u_{2n+1}),$$

$$B_{4,1} = \sum_{i=n+1}^{2n+1} \sum_{n<s\le 2n+1, s\neq i} (-1)^{i+s+n+\delta(s>i)}.$$

$$u_s \cdot u_i \cdot \omega(\omega(u_1, \dots, u_n), u_{n+1}, \dots, \hat{u_s}, \dots, \hat{u_i}, \dots, u_{2n+1}),$$

$$B_{4,2} = \sum_{i=n+1}^{2n+1} \sum_{n<s\le 2n+1, s\neq i} (-1)^{i+s+n+\delta(s>i)}.$$

$$u_s \cdot \omega(u_1, \dots, u_n) \cdot \omega(u_i, u_{n+1}, \dots, \hat{u_s}, \dots, \hat{u_i}, \dots, u_{2n+1}).$$

We see that

$$B_{4,2} =$$

$$\omega(u_1, \dots, u_n) \cdot$$

$$\sum_{i=n+1}^{2n+1} \sum_{n<s\le 2n+1} (-1)^s u_s \cdot \omega(u_{n+1}, \dots, \hat{u_s}, \dots, u_{2n+1}) =$$

$$n\,\omega(u_1, \dots, u_n) \cdot \sum_{n<s\le 2n+1} (-1)^s u_s \cdot \omega(u_{n+1}, \dots, \hat{u_s}, \dots, u_{2n+1}).$$

Further,

$$B_{2,2} = \sum_{i=n+1}^{2n+1} \sum_{j=1}^{n} \sum_{n<s\leq 2n+1, s\neq i} (-1)^{i+j+n+1}.$$

$$u_s \cdot \omega(u_1, \ldots, \hat{u_j}, \ldots, u_n, u_i) \cdot \omega(u_{n+1}, \ldots, u_{s-1}, u_j, u_{s+1}, \ldots, \hat{u_i}, \ldots, u_{2n+1}).$$

Therefore,

$$B_{2,2} + B_{4,2} =$$

$$\pm n \sum_{s=n+1}^{2n+1} u_s \cdot f^{\omega}(u_1, \ldots, u_{n-1}, u_n, \ldots, \hat{u_s}, \ldots, u_{2n+1})$$

So, if $n \equiv 0(mod\, p)$, or $f^{\omega} = 0$, then

$$B_{2,2} + B_{4,2} = 0.$$

Let

$$A_2 - B_{2,1} - B_{4,1} = \sum_{1\leq i<s\leq 2n+1} u_i \cdot u_s \cdot g_{i,s},$$

where the polynomial $g_{i,s}$ does not depend from u_i and u_s. One can check that $g_{i,s} = 0$, if $1 \leq i, s \leq n$ or $n < i, s \leq 2n+1$ By n-Lie identity for ω we have $g_{i,s} = 0$, if $1 \leq i \leq n, n < s \leq 2n+1$.

So, $\tilde{\omega}$ satisfies $(n+1)$-Lie identity. □

COROLLARY 6.4. *Let U be an associative commutative algebra with commuting derivations $\partial_1, \ldots, \partial_n$. Let $\tilde{\omega} = id \wedge \partial_1 \wedge \cdots \wedge \partial_n$ Then $(U, \cdot, \tilde{\omega})$ is $(n+1)$-Lie with identity $f^{\tilde{\omega}} = 0$. If U has unit 1, then the algebra $(U, \cdot, \tilde{\omega})$ satisfies the identity* (4).

The algebra $(U, \cdot, \tilde{\omega})$ constructed in corollary (6.4) is called *n-Lie Jacobian algebra of type W*.

PROOF. Follows from corollary 6.2 and theorem 6.3. □

7. Polynomial principle and $\mathcal{D}$-invariants

In this section we discuss two methods used in our calculations.

We call the first method the *polynomial principle.* It means the following. Let U be some associative commutative algebra over a field K with binary multiplication $U \times U \to U, (u, v) \mapsto u \cdot v$. Let $\partial_1, \ldots, \partial_n$ be commuting derivations of $U : [\partial_i, \partial_j] = 0$, for any $i, j = 1, \ldots, n$. Let there be given a polylinear map $\omega : U \times \cdots \times U \to U$ with n arguments, such that $\omega(u_1, \ldots, u_n)$ is a linear combination of elements of the form $\partial_1^{i_1}(u_1) \cdot \cdots \cdot \partial_n^{i_n} u_n$ for any $u_1, \ldots, u_n \in U$.

Suppose that some statement concerning associative commutative algebra U, n-ary polylinear map $\omega : U \times \cdots \times U \to U$ and derivations $\partial_1, \ldots, \partial_n$ was obtained by using:

- Leibniz rule:

$$\partial_i(u_1 \cdot u_2) = \partial_i(u_1) \cdot u_2 + u_1 \cdot \partial_i(u_2),$$

  for any $u_1, u_2 \in U$ and $i = 1, \ldots, n$.
- linear properties of U
- associativity and commutativity properties of U.

Then this statement is true for any associative commutative algebra $\tilde{U}$ with commuting derivations $\tilde{\partial}_1, \ldots, \tilde{\partial}_n$. In particular, this statement is true for the polynomial algebra $U = K[x_1, \ldots, x_n]$ with derivations $\partial_i = \partial/\partial x_i, i = 1, \ldots, n$.

The second method is based on the study of $\mathcal{D}$-invariants [3]. Let $\mathcal{D}$ be an abelian subalgebra of $Der\, U$ generated by commuting derivations $\partial_1, \ldots, \partial_n$. Let $C^k(U,U) = \{\psi : U \times \cdots \times U \to U\}$ be the space of skew-symmetric polylinear maps with k arguments and $C^*(U,U) = \oplus_k C^k(U,U)$.

Let $\rho : \mathcal{D} \to End\, C^*(U,U)$ be a representation of an abelian Lie algebra $\mathcal{D}$ defined by

$$\rho(X)\psi(u_1, \ldots, u_k) =$$

$$X(\psi(u_1, \ldots, u_k)) - \sum_{l=1}^{n} \psi(u_1, \ldots, u_{l-1}, X(u_l), u_{l+1}, \ldots, u_k),$$

for $\psi \in C^k(U,U)$. We say that $\psi \in C^*(U,U)$ is *$\mathcal{D}$-invariant,* if $\rho(X)\psi = 0$, for any $X \in \mathcal{D}$.

Define the $\wedge$ and $\wedge'$ products on $C^*(U,U)$. For $\psi \in C^k(U,U)$ and $\phi \in C^l(U,U)$ set

$$\psi \smile \phi(u_1, \ldots, u_{k+l}) = \psi(u_1, \ldots, u_k) \cdot \phi(u_{k+1}, \ldots, u_{k+l}),$$

$$\psi \smile' \phi(u_1, \ldots, u_{k+l}) = \psi(\phi(u_1, \ldots, u_l), u_{l+1}, \ldots, u_{k+l}).$$

Define $\psi \wedge \phi \in C^{k+l}(U,U)$ and $\psi \wedge' \phi \in C^{k+l-1}(U,U)$ by

$$\psi \wedge \phi(u_1, \ldots, u_{k+l}) =$$

$$\sum_{\sigma \in Sym_{k,l}} sign\, \sigma\, (\psi \smile \phi)(u_{\sigma(1)}, \ldots, u_{\sigma(k)}, u_{\sigma(k+1)}, \ldots, u_{\sigma(k+l)}),$$

$$\psi \wedge' \phi(u_1, \ldots, u_{k+l}) =$$

$$\sum_{\sigma \in Sym_{l,k-1}} sign\, \sigma\, (\psi \smile' \phi)(u_{\sigma(1)}, \ldots, u_{\sigma(l)}, u_{\sigma(l+1)}, \ldots, u_{\sigma(k+l)}).$$

PROPOSITION 7.1. *If $\psi, \phi \in C^*(U,U)$, then*

$$\rho(X)(\psi + \phi) = \rho(X)\psi + \rho(X)\phi,$$

$$\rho(X)(\lambda\psi) = \lambda\rho(X)\psi,$$

$$\rho(X)(\psi \wedge \phi) = \rho(X)\psi \wedge \phi + \psi \wedge \rho(X)\phi,$$

$$\rho(X)(\psi \wedge' \phi) = \rho(X)\psi \wedge' \phi + \psi \wedge' \rho(X)\phi.$$

COROLLARY 7.1. *The subspace of $\mathcal{D}$-invariants $C^k(U,U)^{\mathcal{D}} = \{\psi \in C^k(U,U) : \rho(X)\psi = 0, \forall X \in \mathcal{D}\}$ is close under $\wedge$ and $\wedge'$ products.*

So, any polylinear map constructed by $\mathcal{D}$-invariant maps using $\wedge$ and $\wedge'$ products will be also $\mathcal{D}$-invariant. For example, $Jac_n^S \in C^n(U,U)^{\mathcal{D}}$ and $Jac_{n+1}^W \in C^{n+1}(U,U)^{\mathcal{D}}$.

Let $pr : U \to K$ be a projection map:

$$pr \sum_{\alpha \in \mathbf{Z}^n} \lambda_\alpha x^\alpha = \lambda_0.$$

Prolong this map to

$$pr : C^*(U,U) \to C^*(U,K),$$

$$(pr\, \psi)(u_1, \ldots, u_n) = pr(\psi(u_1, \ldots, u_n)).$$

THEOREM 7.2. *Let* $\psi \in C^k(U,U)^{\mathcal{D}}$. *Then*

$$\psi(u_1,\dots,u_k) = \sum_{\alpha(1),\dots,\alpha(n)\in \mathbf{Z}_+^n} \frac{\partial^{\alpha(1)}(u_1)}{\alpha(1)!}\cdots\frac{\partial^{\alpha(k)}(u_k)}{\alpha(k)!} pr\,\psi(x^{\alpha(1)},\dots,x^{\alpha(k)}).$$

This theorem follows from the results of [3]. Call k-tuples $(x^{\alpha(1)},\dots,x^{\alpha(k)})$, or simply $(\alpha(1),\dots\alpha(k))$, such that $pr\,\psi(x^{\alpha(1)},\dots,x^{\alpha(k)}) \neq 0$, as a *support* of ψ. So, by theorem 7.2 to prove $\psi = 0$ it is enough to establish that ψ does not have any support k-tuple.

8. Derivations of Jacobian algebras of type S

In this section *char* $K = 0$ and $U = K_n$ or K_n^+ and $n > 1$. if otherwise is not stated. Elements of U are denoted as $u, v, w, u_1, v_1, w_1, \dots$. Let $\theta = \sum_{i=1}^n \epsilon_i$, where $\epsilon_i = (0,\dots,0,\underset{i}{1},0,\dots,0) \in \mathbf{Z}^n$.

There are four Cartan Type Lie algebras of formal vector fields. One of them is called Special. This algebra is defined as the algebra of divergence-free vector fields

$$S_{n-1} = S_{n-1}(U) =< X = u_i\partial_i : Div\,X = \sum_{i=1}^n \partial_i(u_i) = 0 > .$$

Recall that $S_n(K_n^+)$ is simple, but $L = S_{n-1}(K_n)$ is not simple: its commutant $[L,L]$ is generated by derivations $D_{ij} = \partial_i(u)\partial_j - \partial_j(u)\partial_i$, $u \in U$ and

$$L/[L,L] =< D_{ij}x^{-\theta}, x^{-\theta+\epsilon_i}\partial_i : i = 1,\dots,n >\cong K^{n+1}.$$

THEOREM 8.1. *Let* $U = K_n$.

i) *The algebra* (U, Jac_n^S) *is not simple. It has* 1*-dimensional center* $< 1 >$ *and the ideal of codimension 1:* $\bar{U} =< x^\alpha : \alpha \neq -\theta >$. *The factor-algebra* $(\bar{U}/ < 1 >, Jac_n^S)$ *is a simple* n*-Lie algebra.*

ii) *There is an exact sequence*

$$0 \to Out\,(U,\cdot,Jac_n^S) \to Out\,(U,Jac_n^S) \to K^2 \to 0.$$

More exactly, the factor-algebra $Out\,(U,Jac_n^S)/Out\,(U,\cdot,Jac_n^S)$ *is* 2*-dimensional and is generated by derivations* Δ *and* $D_{-\theta}$, *defined by*

$$\Delta = \sum_{i=1}^n x_i\partial_i + n(1-n)^{-1} : x^\alpha \mapsto (|\alpha| + n(1-n)^{-1})x^\alpha,$$

$$D_{-\theta} : x^\alpha \mapsto \delta_{\alpha,-\theta}.$$

iii) *The linear maps* $D_i : U \to U, i = 1,\dots,n$, *given by*

$$D_i : u \mapsto x^{-\theta+\epsilon_i}\partial_i(u),$$

are outer derivations of $(U,\cdot,Jac_n^S)$. *Their classes of derivations in* $Out\,(U,\cdot,Jac_n^S) = Der\,(U,Jac_n^S)/Int\,(U,Jac_n^S)$ *form a base. In particular,* $dim\,Out\,(U,\cdot,Jac_n^S) = n$ *and* $dim\,Out\,(U,Jac_n^S) = n+2$.

iv) *Classes of derivations* $\Delta, D_1,\dots,D_n$ *in* $Out\,(\bar{U},Jac_n^S)/ < 1 >)$ *form a base. In particular,* $dim\,Out\,(\bar{U},Jac_n^S)/ < 1 >= n+1$.

COROLLARY 8.2. *Let* $U = K[x_1^{\pm\frac{1}{n-1}}, \dots, x_n^{\pm\frac{1}{n-1}}]$. *Then eigenspaces of the derivation* Δ *endow the algebra* (U, Jac_n^S) *by grading. If*

$$U_{[k]} = \{u \in U : \Delta(u) = k\,u\} = \{x^\alpha : |\alpha| = k + n(n-1)^{-1}\},$$

then

$$u_1 \in U_{[k_1]}, \dots, u_n \in U_{[k_n]} \Rightarrow Jac_n^S(u_1, \dots, u_n) \in U_{[k_1+\dots+k_n]}.$$

Here $k \in (n-1)^{-1}\mathbf{Z}$.

THEOREM 8.3. *Let* $U = K_n^+$. *Then the algebra* (U^+, Jac_n^S) *has* 1*-dimensional center* $< 1 >$. *Its factor-algebra* $(U^+/ < 1 >, Jac_n^S)$ *is simple* n*-Lie. It has one outer derivation* Δ. *In particular,* $\dim Out\,(U, Jac_n^S) =$
$\dim Out\,(U/ < 1 >, Jac_n^S) = 1$.

Theorem 8.3 follows from theorem 8.1. Therefore, below we can consider only the case $U = K_n$.

LEMMA 8.1. *Any divergence-free derivation of* U *is a linear combination of derivations* $D_{i,j}(u)$ *and derivations* $x^{-\theta+\epsilon_i}\partial_i$.

LEMMA 8.2. *Any derivation of the form* $D_{i,j}(u)$, *where* $i < j, i, j = 1, \dots, n, u \in U$, *can be presented as an interior derivation* $L_{u_1,\dots,u_{n-1}}$ *of the Jacobian algebra* Jac_n^S.

PROOF. It is easy to see that

$$Jac_n^S(x_1, \dots, x_{i-1}, u, x_{i+1}, \dots, x_{j-1}, v, x_{j+1}, \dots, x_n) =$$
$$\partial_i(u)\partial_j(v) - \partial_j(u)\partial_i(v).$$

Therefore,

$$D_{i,j}(u) = (-1)^{n-j} ad\,(x_1, \dots, x_{i-1}, u, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n).$$

□

LEMMA 8.3. *The element* $x^{-\theta}$ *cannot be presented as a Jacobian* $Jac_n^S(u_1, \dots, u_n)$. *Any element* x^β, *where* $\beta \in \mathbf{Z}^n, \beta \neq -\theta$, *can be presented as a linear combination of elements* $Jac_n^S(u_1, \dots, u_n)$, *for some* $u_1, \dots, u_n \in U$.

In other words, the first cohomology group of (U, Jac_n^S) with coefficients in the trivial module is 1-dimensional.

PROOF. Since $Jac_n^S : \wedge^n U \to U$ is polylinear, if $x^{-\theta} = Jac_n^S(u_1, \dots, u_n)$, for some $u_1, \dots, u_n \in U$, then

$$\lambda x^{-\theta} = Jac_n^S(x^{\alpha_1}, \dots, x^{\alpha_n}),$$

for some $\lambda \in K, \alpha_1, \dots, \alpha_n \in \mathbf{Z}^n$. Thus,

$$\lambda = x^\theta Jac_n^S(x^{\alpha_1}, \dots, x^{\alpha_n}) =$$
$$\sum_{\sigma \in Sym_n} sign\,\sigma\, x_{\sigma(1)}\partial_{\sigma(1)}(x^{\alpha_1}) \cdots x_{\sigma(n)}\partial_{\sigma(n)}(x^{\alpha_n}) =$$
$$(\sum_{\sigma \in Sym_n} sign\,\sigma\, \alpha_{1,\sigma(1)} \cdots \alpha_{n,\sigma(n)}) x^{\alpha_1+\dots+\alpha_n},$$

where $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,n}) \in \mathbf{Z}^n$. Therefore, we come to two conclusions: the first

$$\alpha_1 + \dots + \alpha_n = 0 \in \mathbf{Z}^n,$$

and the second,

$$\lambda = \det(\alpha_{i,j}).$$

The first relation means that the sum of rows of the matrix $(\alpha_{i,j})$ is 0. Therefore, its determinant is also 0 and by the second relation $\lambda = 0$.

Let $\beta \neq -\theta$. Then $\beta_i \neq -1$ for some i. Therefore, $\beta_i + 1 \neq 0$, and

$$x^\beta = Jac_n^S(x_1, \ldots, x_{i-1}, (\beta_i + 1)^{-1} x^{\beta+\epsilon_i}, x_{i+1}, \ldots, x_n).$$

□

COROLLARY 8.4. $D_{-\theta} \in Der\,(U, Jac_n^S)$.

LEMMA 8.4. *The element $x^{-\theta}$ cannot be presented in the form $x^{-\theta+\epsilon_i}\partial_i(u)$, where $u \in U, i = 1, \ldots, n$.*

PROOF. Suppose that $\lambda x^{-\theta} = x^{-\theta+\epsilon_i}\partial_i(x^\alpha)$, for some $\alpha \in \mathbf{Z}^n$ and $\lambda \in K$. We have

$$x^{-\theta} = x^{-\theta+\epsilon_i}\partial_i(x^\alpha) = \alpha_i x^{-\theta+\alpha}.$$

Thus $\alpha = 0$, and in particular, $\alpha_i = 0$. So, $\lambda = 0$. □

LEMMA 8.5. *The linear maps $D_i = x^{-\theta+\epsilon_i}\partial_i : u \mapsto x^{-\theta+\epsilon_i}\partial_i(u)$ are derivations of (U, Jac_n^S).*

PROOF. Take $T_{i,j} = x^{-\theta+\epsilon_i+\epsilon_j} \ln x_j, i < j$. Then $T_{i,j} \notin U = K_n$, but $T_{i,j} \in \tilde{U}$, where $\tilde{U} = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, T_{i,j}]$ is an extension of U by $T_{i,j}$. We see that $D_i = -D_{i,j}(T_{i,j})$. So, by lemma 8.2, D_i is a derivation of $Jac_n^S(\tilde{U})$. In particular, D_i satisfies the Leibniz rule

$$D_i Jac_n^S(u_1, \ldots, u_n) = \sum_{l=1}^{n} Jac_n^S(u_1, \ldots, u_{l-1}, D_i(u_l), u_{l+1}, \ldots, u_n),$$

for elements $u_1, \ldots, u_n$ of the subalgebra $(U, Jac_n^S) \subset (\tilde{U}, Jac_n^S)$. Since $D_i(u) \in U$, for any $u \in U$, this means that $D_i \in Der\,(U, J_n^S)$. □

LEMMA 8.6. *The linear map $\Delta : U \to U$ given by*

$$\Delta = \sum_{i=1}^{n} x_i \partial_i + n(1-n)^{-1}$$

is a derivation of the algebra (U, Jac_n^S).

PROOF. Define polynomial f by

$$f(u_1, \ldots, u_n) = \Delta(Jac_n^S(u_1, \ldots, u_n)) - \sum_{l=1}^{n} Jac_n^S(u_1, \ldots, u_{l-1}, \Delta(u_l), u_{l+1}, \ldots, u_n).$$

Notice that $[\partial_i, \Delta] = \partial_i$. Since Jac_n^S is $\mathcal{D}$-invariant,

$$\rho(\partial_i) f = \rho([\partial_i, \Delta]) Jac_n^S = \rho(\partial_i) Jac_n S = 0.$$

So, f is $\mathcal{D}$-invariant and by theorem 7.2,

$$f(u_1, \ldots, u_n) = \sum_{\alpha(1), \ldots, \alpha(n) \in \mathbf{Z}_+^n} \frac{\partial^{\alpha(1)}(u_1)}{\alpha(1)!} \cdots \frac{\partial^{\alpha(n)}(u_n)}{\alpha(n)!} pr\, f(x^{\alpha(1)}, \ldots, x^{\alpha(n)}),$$

where $|\alpha(1)| + \cdots + |\alpha(n)| = n$.

We see that $pr\, f : \wedge^n U \to K$ is uniquely defined by $f(x_1, \dots, x_n)$. In other words,

$$Support(f) \subseteq C^n(R(\pi_1), K) \cong < x_1 \wedge \cdots \wedge x_n > \cong K.$$

We have,

$$\Delta(1) = n(1-n)^{-1}, \quad \Delta(x_i) = (1-n)^{-1}, \quad Jac_n^S(x_1, \dots, x_n) = 1.$$

Therefore,

$$f(x_1, \dots, x_n) = n(1-n)^{-1} - \sum_{l=1}^{n} (1-n)^{-1} = 0.$$

So, $Support(f) = \emptyset$, and $pr\, f = 0$. Thus, by theorem 7.2, $f = 0$. In other words, $\Delta \in Der\,(U, Jac_n^S)$. □

LEMMA 8.7. *Derivations $\Delta, D_1, \dots, D_n \in Der\,(U, Jac_n^S)$ are outer. Moreover, any non-trivial linear combination of derivations $D_1, \dots, D_n$ and Δ is outer.*

PROOF. Suppose that

$$\lambda_0 \Delta(v) + \sum_{i=1}^{n} \lambda_i D_i(v) = Jac_n^S(u_1, \dots, u_{n-1}, v),$$

for some $\lambda_0, \lambda_1, \dots, \lambda_n \in K, u_1, \dots, u_{n-1} \in U$ and for any $v \in U$.

For $v = 1$ we obtain that $n(1-n)^{-1}\lambda_0 = 0$, and $\lambda_0 = 0$.

For $v = x_i$ we have

$$\lambda_i x^{-\theta+\epsilon_i} = (-1)^{n-i} Jac_{n-1}^S(u_1, \dots, u_{n-1}).$$

By lemma 8.3, this is possible only for the case of $\lambda_i = 0$. □

LEMMA 8.8. *Suppose that $D_{i,j}(x^\beta)u = 0, u \in U$, for any $\beta \in \mathbf{Z}_+^n, |\beta| > 2$. Then $u \in < 1 >$.*

PROOF. Suppose that $u = \sum_{\alpha \in \mathbf{Z}^n, q \le |\alpha| \le s} \lambda_\alpha x^\alpha$ and $\lambda_{\bar\alpha} \neq 0$, for some $\bar\alpha \in \mathbf{Z}^n$, such that $|\bar\alpha| = s$ and $\bar\alpha \neq 0$. Then $\bar\alpha_i \neq 0$, for some i. Take $\beta = 3\epsilon_j$, for some $j \neq i$. Recall that we suppose that $n > 1$. Then $D_{i,j}(x^\beta) = -3x_j^2 \partial_i$ and

$$D_{i,j}(x^\beta)u = -3\lambda_{\bar\alpha}\bar\alpha_i x^{\bar\alpha + 2\epsilon_j} + w,$$

for some $w \in U$. Notice that w is a linear combination of x^γ, where $\gamma \in \mathbf{Z}^n, |\gamma| \le s+2$ and $\gamma \neq \bar\alpha + 2\epsilon_j$. Therefore, $-3\lambda_{\bar\alpha}\bar\alpha_i = 0$, i.e., $\lambda_{\bar\alpha_i} = 0$. The obtained contradiction shows that $s = 0$. By similar arguments it is easy to obtain that $q = 0$. □

LEMMA 8.9. *Let $D \in Der\,(U, Jac_n^S)$, and $D(u) = 0$, for any $u \in U^+$. Then $D = \lambda D_{-\theta}$, for some $\lambda \in K$.*

PROOF. Suppose that $D(x^\alpha) \neq 0$, for some $\alpha \in \mathbf{Z}^n, \alpha \notin \mathbf{Z}_+^n$ and $D(x^\beta) = 0$, for any $\beta \in \mathbf{Z}^n, |\beta| > |\alpha|$. Notice that

$$Jac_n^S(x_1, \dots, x_{i-1}, u, x_{i+1}, \dots, x_{j-1}, v, x_{j+1}, \dots, x_n) = D_{i,j}(u)v = -D_{i,j}(v)u.$$

For any $\beta \in \mathbf{Z}^n, |\beta| > 2$,

$$|Jac_n^S(x_1, \dots, x_{i-1}, x^\alpha, x_{i+1}, \dots, x_{j-1}, x^\beta, x_{j+1}, \dots, x_n)| > |x^\alpha|.$$

So,

$$D(Jac_n^S(x_1, \dots, x_{i-1}, x^\alpha, x_{i+1}, \dots, x_{j-1}, x^\beta, x_{j+1}, \dots, x_n)) = 0,$$

if $\beta \in \mathbf{Z}_+^n, |\beta| > 2$. Since D is derivation, and $D(u) = 0$, for any $u \in U^+$,

$$D(Jac_n^S(x_1, \ldots, x_{i-1}, x^\alpha, x_{i+1}, \ldots, x_{j-1}, x^\beta, x_{j+1}, \ldots, x_n)) =$$

$$Jac_n^S(x_1, \ldots, x_{i-1}, D(x^\alpha), x_{i+1}, \ldots, x_{j-1}, x^\beta, x_{j+1}, \ldots, x_n)),$$

Thus,

$$D_{i,j}(x^\beta)(D(x^\alpha)) = 0,$$

for any $\beta \in \mathbf{Z}_+^n, |\beta| > 2$. By lemma 8.8,

$$D(x^\alpha) \in \cap_{s=1}^n Ker\partial_s =< 1 > .$$

So,

$$D(x^\alpha) = \lambda \cdot 1, \tag{6}$$

for some $\lambda \in K$.

Suppose that $\lambda \neq 0$.

Now take $\beta = \epsilon_j$. Then by the Leibniz rule

$$D(\partial_i(x^\alpha)) = D(Jac_n^S(x_1, \ldots, x_{i-1}, x^\alpha, x_{i+1}, \ldots, x_n)) =$$

$$Jac_n^S(x_1, \ldots, x_{i-1}, D(x^\alpha), x_{i+1}, \ldots, x_n)) = \partial_i(D(x^\alpha)),$$

and, according to (6),

$$D(\partial_i(x^\alpha)) = 0, \tag{7}$$

for any $i = 1, \ldots, n$.

We have

$$Jac_n^S(x_1, \ldots, x_{i-1}, x^\alpha, x_{i+1}, \ldots, x_{j-1}, x_i x_j, x_{j+1}, \ldots, x_n) =$$

$$x_i \partial_i(x^\alpha) - x_j \partial_j(x^\alpha).$$

Therefore, by (6) and the Leibniz rule,

$$(\alpha_i - \alpha_j) D(x^\alpha) =$$

$$D(Jac_n^S(x_1, \ldots, x_{i-1}, x^\alpha, x_{i+1}, \ldots, x_{j-1}, x_i x_j, x_{j+1}, \ldots, x_n)) = 0.$$

So, if $\lambda \neq 0$, then

$$\alpha_i = t, \tag{8}$$

for some $0 \neq t \in \mathbf{Z}$ and for any $i = 1, \ldots, n$.

Take $\beta = \epsilon_i + 2\epsilon_j$. We have

$$Jac_n^S(x_1, \ldots, x_{i-1}, \partial_j(x^\alpha), x_{i+1}, \ldots, x_{j-1}, x_i x_j^2, x_{j+1}, \ldots, x_n) = (2\alpha_i - \alpha_j + 1)x^\alpha.$$

Then by (6), (8) we obtain that

$$(2\alpha_i - \alpha_j + 1) D(x^\alpha) =$$

$$D(Jac_n^S(x_1, \ldots, x_{i-1}, \partial_j(x^\alpha), x_{i+1}, \ldots, x_{j-1}, x_i x_j^2, x_{j+1}, \ldots, x_n)) = 0.$$

So, by (8), $t = -1$ and $D = \lambda D_{-\theta}$. □

LEMMA 8.10. *Any derivation of (U, Jac_n^S) is a linear combination of some interior derivation and derivations $\Delta, D_{-\theta}, D_1, \ldots, D_n$.*

PROOF. By lemma 8.9 any derivation D is defined by its restriction on U^+ up to $D_{-\theta}$. Any linear map $F : U^+ \to U$ is a linear combination of linear maps of the form $x^\alpha\partial^\beta : U \to U, v \mapsto x^\alpha(\partial^\beta(v))$, where $\alpha \in \mathbf{Z}^n, \beta \in \mathbf{Z}_+^n$. Suppose that $F = \sum_{\alpha\in\mathbf{Z}^n,\beta\in\mathbf{Z}_+^n} \lambda_{\alpha,\beta}x^\alpha\partial^\beta \in Der\,(U, J_n^S)$.

Step 1. Any derivation F is a linear combination of Δ and some derivation F' with the property $F'(1) = 0$. Let us prove it. Since 1 is central, $F(1)$ is also central for any derivation F. The center of U is 1-dimensional and is generated by one element 1. Therefore, $F(1) = \lambda_{0,0}$. If $\lambda_{0,0} \neq 0$, we can take instead of F

$$F' = F - \lambda_{0,0}(1-n)n^{-1}\Delta.$$

Then $F' \in Der\,(U, Jac_n^S)$ and $\lambda'_{0,0} = 0$, if

$$F' = \sum_{\alpha\in\mathbf{Z}^n,\beta\in\mathbf{Z}_+^n} \lambda'_{\alpha,\beta}x^\alpha\partial^\beta.$$

Step 2. Suppose that the derivation F' satisfies the following normalization

$$F'(1) = 0 \tag{9}$$

Prove that F' can be presented as a linear combination of derivations $D_1, \ldots, D_n$ and some interior derivation and some differential operator of differential order > 1.

Take

$$\bar{F}' = F' - \sum_{i=1}^{n} \lambda'_{-\theta+\epsilon_i,\epsilon_i} D_i.$$

Then $F'' \in Der\,(U, Jac_n^S)$ and

$$\bar{F}'(x_i) \in < x^\alpha : \alpha \neq -\theta + \epsilon_i >,$$

for any $i = 1, \ldots, n$. Let $\bar{F}' = \sum_{\alpha\in\mathbf{Z}^n,\beta\in\mathbf{Z}_+^n} \bar{\lambda}'_{\alpha,\beta}x^\alpha\partial^\beta$. Then

$$\bar{\lambda}_{\alpha,\epsilon_1} \neq 0 \Rightarrow \alpha_j \neq -1, \text{ for some } j.$$

Take

$$F'' = \bar{F}' + \bar{\lambda}_{\alpha,\epsilon_1}(\alpha_j + 1)^{-1}D_{1,j}(x^{\alpha+\epsilon_j}).$$

It is easy to see that $F'' \in Der\,(U, Jac_n^S)$, and

$$\lambda''_{\alpha,\epsilon_1} = 0,$$

where $F'' = \sum_{\alpha\in\mathbf{Z}^n,\beta\in\mathbf{Z}_+^n} \lambda''_{\alpha,\beta}x^\alpha\partial^\beta$.

Let us use induction on l and suppose that F'' some $l > 1$ has the following property: $\lambda''_{\alpha,\epsilon_i} = 0$, for any $\alpha \in \mathbf{Z}^n$, if $i < l$ and $\lambda''_{\alpha(l),\epsilon_l} \neq 0$ for some $\alpha(l) \in \mathbf{Z}^n$.

Let $(\tilde{U}, Jac_{n-l+1}^S)$ be the Jacobian algebra of type S with vector space $\tilde{U} = K[x_l^{\pm1}, \ldots, x_n^{\pm1}]$ and multiplication $Jac_{n-l+1}^S = \partial_l \wedge \cdots \wedge \partial_n$.

Let G be the restriction of F'' to subalgebra $(\tilde{U}, Jac_{n-l+1}^S)$. Then

$$G(u) = \sum_{\alpha'=(\alpha_1,\ldots,\alpha_{l-1})\in\mathbf{Z}^{l-1}} x^{\alpha'}G_{\alpha'}(u),$$

for any $u \in \tilde{U}$, where $G_{\alpha'}$ is a linear map defined on $\tilde{U}$ with coefficients in $\tilde{U}$.

Since $F''(x_i) = 0, i < l$ and $Jac_{n-l+1}^S = i(x_1)\cdots i(x_{l-1})Jac_n^S$, then

$$G(Jac_{n-l+1}^S(u_l, \ldots, u_n)) =$$

$$F''Jac_n^S(x_1, \ldots, x_{l-1}, u_l, \ldots, u_n) =$$

$$\sum_{s=l}^{n} Jac_n^S(x_1,\dots,x_{l-1},u_l,\dots,u_{s-1},F''(u_s),u_{s+1},\dots,u_n) =$$

$$\sum_{s=l}^{n} Jac_{n-l+1}(u_l,\dots,u_{s-1},G(u_s),u_{s+1},\dots,u_n),$$

for any $u_l,\dots,u_n \in < x^\alpha : \alpha \in \mathbf{Z}^n, \alpha_i = 0, i < l >$. Therefore,

$$\sum_{\alpha' \in \mathbf{Z}^{l-1}} x^{\alpha'} G_{\alpha'}(Jac_{n-l+1}(u_l,\dots,u_n)) =$$

$$\sum_{s=1}^{n} \sum_{\alpha' \in \mathbf{Z}^{l-1}} x^{\alpha'} Jac_{n-l+1}(u_l,\dots,u_{s-1},G_{\alpha'}(u_s),u_{s+1},\dots,u_n) =$$

$$\sum_{\alpha' \in \mathbf{Z}^{l-1}} x^{\alpha'} \sum_{s=1}^{n} Jac_{n-l+1}(u_l,\dots,u_{s-1},G_{\alpha'}(u_s),u_{s+1},\dots,u_n).$$

In other words, $G_{\alpha'} \in Der\,(\tilde{U}, J^S_{n-l+1})$, for any $\alpha' \in \mathbf{Z}^n, \alpha'_l = \ldots = \alpha'_n = 0$. By an inductive suggestion $G_{\alpha'}$ is a linear combination of derivations of $(\tilde{U}, Jac^S_{n-l+1}$ of the form $x^{-\sum_{s=l}^n \epsilon_s + \epsilon_j}\partial_j$, where $j = l,\dots,n$ and a derivation of the form $L_{u_l,\dots,u_{n-1}}$, for some $u_l,\dots,u_{n-1} \in \tilde{U}$. Here the derivation $\tilde{\Delta} = \sum_{s=1}^n x_s\partial_s + (n-l+1)/(l-n)$ does not appear because of normalization (9). Thus, $x^{\alpha'}G_{\alpha'}$ is a linear combination of derivations $A_i := x^{\alpha' - \sum_{s=1}^n \epsilon_s + \epsilon_i}\partial_i, i \geq l$, and some linear operator $B_{\alpha'} := x^{\alpha'} L_{u_l,\dots,u_{n-1}}$. Notice that $Div\, A_i = 0, i \geq l$. Moreover, $A_i \in Int\,(U, Jac^S_n)$, because of $\lambda''_{-\theta+\epsilon_i,\epsilon_i} = 0$.

By an inductive suggestion and by lemma 8.1, the divergence-free derivation $L_{u_l,\dots,u_{n-1}}$ is a linear combination of derivations of the form $D_{i,j}(u), l \leq i < j, u \in \tilde{U}$. Therefore, $B_{\alpha'}$, where $\alpha' = (\alpha_1,\dots,\alpha_{l-1},0,\dots,0) \in \mathbf{Z}^n$, is also a linear combination of derivations $D_{i,j}(x^{\alpha'}u), l \leq i < j$. In particular, $B_{\alpha'} \in Int\,(U, Jac^S_n)$.

So, we have proved that G is a linear combination of interior derivations. Therefore, we can add to F'' some linear combination of interior derivations and obtain a new derivation $F''' = \sum_{\alpha \in \mathbf{Z}^n, \beta \in \mathbf{Z}^n_+} \lambda'''_{\alpha,\beta} x^\alpha \partial^\beta$, such that $\lambda'''_{\alpha,\epsilon_i} = 0$, for any $\alpha \in \mathbf{Z}^n$, if $i < l+1$.

Thus, induction on l is possible and finally we obtain that F'' is a sum of some interior derivation and some linear operator

$$H = \sum_{\alpha \in \mathbf{Z}^n, \beta \in \mathbf{Z}^n_+, |\beta| > 1} \mu_{\alpha,\beta} x^\alpha \partial^\beta.$$

Step 3. Prove that $H = 0$. Present H in the form $H = \sum_{\beta \in \mathbf{Z}^n_+, |\beta|>1} H_\beta \partial^\beta$, where $H_\beta = \sum_{\alpha \in \mathbf{Z}^n} \lambda_{\alpha,\beta} x^\alpha$. Suppose that $H \neq 0$ and $H_{\bar{\beta}} \neq 0$, for some $\bar{\beta} \in \mathbf{Z}^n_+, |\bar{\beta}| = t > 1$ and $H_\beta = 0$, for any $\beta \in \mathbf{Z}^n_+, |\beta| < t$. We have

$$Jac^S_n(x_1,\dots,x_{i-1},x^{\bar{\beta}},x_{i+1},\dots,x_n) = \partial_i(x^{\bar{\beta}}),$$

$$H(x_i) = 0, \forall i = 1,\dots,n,$$

Thus,

$$H(Jac^S_n(x_1,\dots,x_{i-1},x^{\bar{\beta}},x_{i+1},\dots,x_n)) = H(\partial_i(x^{\bar{\beta}}) = 0.$$

On the other hand, by the Leibniz rule

$$H(Jac^S_n(x_1,\dots,x_{i-1},x^{\bar{\beta}},x_{i+1},\dots,x_n)) =$$

$$Jac_n^S(x_1,\dots,x_{i-1},H(x^{\bar\beta}),x_{i+1},\dots,x_n)=\partial_i(H(x^{\bar\beta})).$$

So, $H(x^{\bar\beta})\in\cap_{i=1}^n Ker\,\partial_i =< 1 >$. Here $\bar\beta$ is any element of $\mathbf{Z}_+^n$, such that $H_{\bar\beta}\neq 0, |\bar\beta|>1$. In particular, $H_{\epsilon_i+\epsilon_j}=0$ or $H_{\epsilon_i+\epsilon_j}\in< 1 >$. Therefore, in any case by the Leibniz rule

$$(\bar\beta_i-\bar\beta_j)H(x^{\bar\beta})=$$
$$H(Jac_n^S(x_1,\dots,x_{i-1},x^{\bar\beta},x_{i+1},\dots,x_{j-1},x_ix_j,x_{j+1},\dots,x_n))=0.$$

So, $\bar\beta_i=q, i=1,\dots,n$, for some $0<q\in\mathbf{Z}$. In other words, $\bar\beta=q\theta$. In particular, $t=|\bar\beta|=nq\geq n$, and, since $n>1$, $t=|\bar\beta|=nq>q$. Thus $q+1\leq t$, and $H(x_i^{q+1})=0$ or $H(x_i^{q+1}\in< 1 >$. We have $J_n^S(x_1^{q+1},\dots,x_n^{q+1})=(q+1)^n x^{q\theta}$, So, by the Leibniz rule,

$$(q+1)^nH(x^{q\theta})=$$
$$H(Jac_n^S(x_1^{q+1},\dots,x_n^{q+1}))=0.$$

Hence, $H_{\bar\beta}=0$. Contradiction. □

PROOF OF THEOREM 8.1. The simplicity of (U,Jac_n^S) was established in [**5**]. Other statements follow from corollary 8.4 and lemmas 8.5, 8.9 8.6, 8.7 and 8.10. □

9. Derivations of Jacobian algebras of type W

In this section K is the field of characteristic 0 and $U=K_n$ or K_n^+. Let

$$W_n=W_n(U)=< u\partial_i : u\in U, i=1,\dots,n >$$

be General Cartan Type Lie algebra. This algebra is also called Witt algebra.

THEOREM 9.1. *The n-Lie algebra (U,Jac_{n+1}^W) is simple. Any derivation of (U,Jac_{n+1}^W) is interior. Moreover, $Int\,(U,Jac_{n+1}^W)$ is isomorphic to Witt algebra W_n.*

On U we have two multiplications Jac_{n+1}^W and Jac_n^S. They are related by

$$Jac_{n+1}^W(u_1,\dots,u_{n+1})=$$
$$(-1)^{n+1}Jac_n^S(u_1,\dots,u_n)u_{n+1}+\sum_{i=1}^n(-1)^{i+n-1}Jac_{n,\hat i}^W(u_1,\dots,u_n)\partial_i(u_{n+1}),$$

where

$$Jac_{n,\hat i}^W=id\wedge\partial_1\wedge\cdots\wedge\hat{\partial_i}\wedge\cdots\wedge\partial_n.$$

Let W_n+U be a semi-direct sum:

$$[X+u,Y+v]=[X,Y]+X(v)-Y(u),$$

for any $X,Y\in W_n, u,v\in U$.

LEMMA 9.1. *The map $W_n\to W_n+U$ given by $X\mapsto X+\lambda Div\,X$ is a monomorphism for any $\lambda\in K$.*

PROOF. The statement is evident, if $\lambda=0$. Suppose that $\lambda\neq 0$.

The divergence map $X\mapsto Div\,X$ is 1-cocycle in $Z^1(W_n,U)$:

$$Div[X,Y]=X(Div\,Y)-Y(Div\,X).$$

Therefore, for $f=id+\lambda\,Div$,

$$f[X,Y]=[X,Y]+\lambda Div[X,Y]=$$
$$[X,Y]+\lambda(X(Div(Y))-Y(Div(X)))=$$

$$[X + \lambda\, Div\, X, Y + \lambda\, Div\, Y] = [f(X), f(Y)].$$

So, f is the homomorphism of Lie algebras.

Suppose that $X \in Ker\, f$. Then

$$X(u) + \lambda(Div\, X)u = 0,$$

for any $u \in U$. Take $u = 1$. Then $\lambda(Div\, X) = 0$. Thus, $X = \sum_{i=1}^{n} \mu_i \partial_i$. Take now $u = x_i$. We have $0 = X(u) = \lambda_i$. So, $X = 0$. □

LEMMA 9.2. *For any associative commutative algebra U with commuting derivations $\partial_1, \ldots, \partial_n$ the algebra (U, Jac_{n+1}^W) is $(n+1)$- Lie and graded:*

$$J_{n+1}^W(U_{[i_1]}, \ldots, U_{[i_k]}) \subseteq U_{[i_1+\ldots+i_k]},$$

where

$$U_{[s]} =< x^\alpha : |\alpha| = s + 1 >, s \in \mathbf{Z}.$$

PROOF. By corollary 6.4 (U, Jac_{n+1}^W) is $(n+1)$-Lie. It is graded:

$$Jac_{n+1}^W(x^{\alpha(0)}, \ldots, x^{\alpha(n)}) \in< x^{\alpha(0)+\cdots+\alpha(n)-\theta} >,$$

and

$$|Jac_{n+1}^W(x^{\alpha(0)}, \ldots, x^{\alpha(n)})| = |\alpha(0)+\cdots+\alpha(n)-\theta| = (|\alpha(0)|-1)+\ldots+(|\alpha(n)|-1)+1.$$

□

LEMMA 9.3. *Any interior derivation of (U, Jac_{n+1}^W) has the form $X - n^{-1} Div\, X$, where $X \in W_n(U)$.*

PROOF. Any interior derivation

$$L_{u_1,\ldots,u_n} : v \mapsto Jac_{n+1}^W(u_1, \ldots, u_n, v)$$

can be presented as a sum of some differential operator of first order and the operator of multiplication to some element of U

$$L_{u_1,\ldots,u_n} = X_{u_1,\ldots,u_n} + R_{u_1,\ldots,u_n},$$

where

$$X_{u_1,\ldots,u_n} = \sum_{i=1}^{n} (-1)^{n+i-1} Jac_{n,\hat{i}}^W(u_1, \ldots, u_n)\partial_i,$$

$$R_{u_1,\ldots,u_n} v = (-1)^{n+1} Jac_n^S(u_1, \ldots, u_n)v,$$

satisfies the relation

$$Div\, X_{u_1,\ldots,u_n} =$$

$$\sum_{i=1}^{n} (-1)^{n+i-1} \partial_i(id \wedge \partial_1 \wedge \cdots \wedge \hat{\partial}_i \wedge \cdots \wedge \partial_n)(u_1, \ldots, u_n)) =$$

$$\sum_{i=1}^{n} (-1)^{n+i-1} (\partial_i \wedge \partial_1 \wedge \cdots \wedge \hat{\partial}_i \wedge \cdots \wedge \partial_n)(u_1, \ldots, u_n)) =$$

$$(-1)n \sum_{i=1}^{n} (\partial_1 \wedge \cdots \wedge \partial_i \wedge \cdots \wedge \partial_n)(u_1, \ldots, u_n)) =$$

$$(-1)^n n\, Jac_n^S(u_1, \ldots, u_n).$$

In other words,

$$Div\, X_{u_1,\ldots,u_{n-1}} + n\, R_{u_1,\ldots,u_{n-1}} = 0.$$

So, any interior derivation is an element of the subspace $\{X - n^{-1}Div\,X : X \in W_n(U)\}$. □

LEMMA 9.4. *Let U be an associative commutative algebra with the space of commuting derivations $\mathcal{D} = \{\partial_1, \ldots, \partial_n\}$. Then for any $X \in W_n$ the linear map $g = g_X : U \to U$ given by $g(u) = X(u) + \lambda\, Div\, X\, u$ is a derivation of the algebra (U, Jac^W_{n+1}), if and only if $\lambda = -n^{-1}$.*

PROOF. Consider the polynomial

$$f(X, u_1, \ldots, u_{n+1}) =$$

$$g_X(Jac^W_{n+1}(u_1, \ldots, u_{n+1})) - \sum_{i=1}^{n+1} Jac^W_{n+1}(u_1, \ldots, u_{i-1}, g_X(u_i), u_{i+1}, \ldots, u_n).$$

We should prove that $f(u_1, \ldots, u_{n+1}) = 0$ for any $u_1, \ldots, u_{n+1} \in U$, if and only if $\lambda = -n^{-1}$.

Notice that $g = g_X$ is $\mathcal{D}$-invariant for any $X \in W_n$:

$$(\rho(\partial_i)g)(u) = \partial_i(g(u)) - g(\partial_i(u)) =$$

$$\partial_i(Xu) + \lambda\partial_i(Div\, X\, u) - X(\partial_i(u)) - \lambda\, Div\, X\, \partial_i(u) =$$

$$\partial_i(X)u + \lambda\, \partial_i(X)\, u = g(\partial_i X)u.$$

We know that Jac^S_{n+1} is $\mathcal{D}$-invariant:

$$\rho(\partial_i)(id \wedge \partial_1 \wedge \cdots \wedge \partial_n) = 0.$$

Hence by corollary 7.1 f as a linear combination of compositions of $\mathcal{D}$-invariant polynomials, is also $\mathcal{D}$-invariant.

As we mentioned before Jac^W_{n+1} is graded. Notice that g has grade degree $|X|$:

$$g(U_{[s]}) \subseteq U_{[s]+|X|}.$$

Therefore, the polynomial f is also graded.

So, we can use theorem 7.2 to prove that $f = 0$. Notice that f is skew-symmetric in n parameters $u_1, \ldots, u_{n+1}$. Therefore, for homogeneous $X \in W_n$ and $u_1, \ldots, u_{n+1} \in U$,

$$f(X, u_1, \ldots, u_{n+1}) \in U_{[0]},$$

if and only if

$$|X| = -1, |u_1| + \ldots + |u_{n+1}| = 0$$

or

$$|X| = 0, |u_l| = -1 \text{ for some } l \text{ and } |u_1| + \cdots \hat{|u_l|} + \cdots + |u_{n+1}| = 0$$

In other words, if

$$(x^{\alpha(0)}\partial_i, x^{\alpha(1)}, \ldots, x^{\alpha(n+1)}) \in Support(f),$$

then

$$|\alpha(0)| = 0, |\alpha(1)| + \cdots + |\alpha(n+1)| = n+1$$

or

$$|\alpha(0)| = 1, |\alpha(l)| = 0, \text{ for some } l \text{ and } |\alpha(1)| + \cdots + \widehat{|\alpha(l)|} + \cdots + |\alpha(n+1)| = n$$

So, we should check that

$$f(\partial_i, x^{\alpha(1)}, \ldots, x^{\alpha(n+1)}) = 0,$$

if $|\alpha(1)| + \cdots + |\alpha(n+1)| = n$, or

$$f(x_j\partial_i, x^{\alpha(1)}, \ldots, x^{\alpha(n+1)}) = 0,$$

if $\alpha(l) = 0$, for some l and

$$|\alpha(1)| + \cdots + |\hat{\alpha(l)}| + \cdots + +|\alpha(n\ +1)| = n, |\alpha(s)| > 0, s \neq l.$$

The first statement is evident: since ∂_i is a derivation of U, $Div\ \partial_i = 0$, and $[\partial_i, \partial_j] = 0$ for all i and j, then $f(\partial_i, x^{\alpha(1)}, \ldots, x^{\alpha(n+1)}) = 0$, for any $\alpha(1), \ldots, \alpha(n+1)$. Since f is skew symmetric in u_i type parameters, in the second case we can take, say, $l = 1$. Then $|\alpha(2)| = \ldots = |\alpha(n+1)| = 1$. So, in the second case we have

$$f(x_j\partial_i, 1, x_1, \ldots, x_n) =$$

$$x_j\partial_i(Jac^W_{n+1}(1, x_1, \ldots, x_n)) + \lambda\,\delta_{i,j}(Jac^W_{n+1}(1, x_1, \ldots, x_n))$$
$$-Jac^W_{n+1}(\lambda\delta_{i,j}1, x_1, \ldots, x_n)$$
$$-\sum_{s=1}^{n} Jac^W_{n+1}(1, x_1, \ldots, x_{s-1}, x_j\partial_i(x_s) + \lambda\,\delta_{i,j}x_s, x_{s+1}, \ldots, x_n) =$$

$$x_j\partial_i(Jac^S_n(x_1, \ldots, x_n)) + \lambda\,\delta_{i,j}(Jac^S_n(x_1, \ldots, x_n))$$
$$-\lambda\,\delta_{i,j}Jac^S_n(x_1, \ldots, x_n)$$
$$-\sum_{s=1}^{n} Jac^S_n(x_1, \ldots, x_{s-1}, x_j\partial_i(x_s) + \lambda\,\delta_{i,j}x_s, x_{s+1}, \ldots, x_n) =$$

$$-\sum_{s=1}^{n} \delta_{i,s}Jac^S_n(x_1, \ldots, x_{s-1}, x_j, x_{s+1}, \ldots, x_n)$$
$$-\sum_{s=1}^{n} \lambda\,\delta_{i,j}Jac^S_n(x_1, \ldots, x_{s-1}, x_s, x_{s+1}, \ldots, x_n) =$$
$$-Jac^S_n(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n) - \lambda\,n\delta_{i,j} =$$

$$-\delta_{i,j}(1 + \lambda n).$$

Thus $f = 0$, iff $\lambda = -n^{-1}$. □

COROLLARY 9.2. *Any divergence-free derivation of U is also a derivation of (U, Jac^W_{n+1}).*

LEMMA 9.5. *Suppose that $D(1) = 0, D(x_i) = 0$ for any $i = 1, \ldots, n$. Then $D(\partial_i(u)) = \partial_i(D(u))$ for any $u \in U$.*

PROOF. Since

$$Jac^W_{n+1}(u, 1, x_1, \ldots, \hat{x_i}, \ldots, x_n) = \partial_i(u),$$

then, by the Leibniz rule

$$D(\partial_i(u)) = D(Jac^W_{n+1}(u, 1, x_1, \ldots, \hat{x_i}, \ldots, x_n)) =$$
$$Jac^W_{n+1}(D(u), 1, x_1, \ldots, \hat{x_i}, \ldots, x_n) = \partial_i(D(u)),$$

for any $u \in U$. □

LEMMA 9.6. *Let $U = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Suppose that $D \in Der\,(U, Jac^W_{n+1})$ and $D(u) = 0$, for any $u \in U^+$. Then $D(u) = 0$, for any $u \in U$.*

PROOF. Suppose that $D(x^\alpha) \neq 0$ for some $\alpha \in \mathbf{Z}^n \setminus \mathbf{Z}_+^n$ and $D(x^\beta) = 0$, for all $\beta \in \mathbf{Z}^n, |\beta| > |\alpha|$. If $\alpha_i + \epsilon_i < 0$, for some i, then we obtain a contradiction: by lemma 9.5 in this case

$$|\alpha + \epsilon_i| > |\alpha| \Rightarrow D(x^{\alpha+\epsilon_i}) = 0 \Rightarrow D(x^\alpha) = (\alpha_i + 1)^{-1} D(\partial_i(x^{\alpha+\epsilon_i})) = 0.$$

Thus, $\alpha_i = -1$, for all $i = 1, \ldots, n$. Notice that

$$Jac_{n+1}^W(u, x_1^2, \ldots, x_n^2) = 2^{n-1} x^\theta (2 - \sum_{i=1}^n x_i \partial_i) u,$$

for any $u \in U$. In particular,

$$Jac_{n+1}^W(x^{-2\theta}, x_1^2, \ldots, x_n^2) = 2^n (n+1) x^{-\theta}.$$

Thus,

$$D(x^{-\theta}) =$$
$$2^{-n}(n+1)^{-1} D(Jac_{n+1}^W(x^{-2\theta}, x_1^2, \ldots, x_n^2)) =$$
$$2^{-n}(n+1)^{-1} Jac_{n+1}^W(D(x^{-2\theta}), x_1^2, \ldots, x_n^2) =$$
$$2^{-1}(n+1)^{-1} x^\theta (2 - \sum_{i=1}^n x_i \partial_i) D(x^{-2\theta}).$$

On the other hand,

$$|x_i^{-1}| > |x^{-\theta}| \Rightarrow D(x_i^{-1}) = 0, i = 1, \ldots, n,$$
$$Jac_{n+1}^W(1, x_1^{-1}, \ldots, x_n^{-1}) = (-1)^n x^{-2\theta},$$

and by the Leibniz rule

$$D(x^{-2\theta}) =$$
$$(-1)^n D(Jac_{n+1}^W(1, x_1^{-1}, \ldots, x_n^{-1})) = 0.$$

Therefore, $D(x^{-\theta}) = 0$. Contradiction. □

LEMMA 9.7. *Any derivation of the algebra* (U, Jac_{n+1}^W) *has the form* $X - n^{-1} Div\, X$ *for some* $X \in W_n(U)$.

PROOF. By lemma 9.6 any derivation of $F \in Der\,(U, Jac_{n+1}^W)$ is defined by its restriction on $U^+ = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let D be the restriction of derivation F on U^+.

Step 1. Prove that

$$D(1) \in < x^\alpha : \alpha \in \mathbf{Z}^n : \alpha \neq -\theta > . \tag{10}$$

Notice that $|x^{-\theta}| = -2n \neq 0$. Therefore, if $\sum_{i=1}^n x_i \partial_i D(1) = 0$, then (10) is true.

Consider now the case $\sum_{i=1}^n x_i \partial_i D(1) \neq 0$. Since $1 = Jac_{n+1}^W(1, x_1, \ldots, x_n)$, by the Leibniz rule we have

$$D(1) =$$
$$Jac_{n+1}^W(D(1), x_1, \ldots, x_n) + \sum_{i=1}^n Jac_{n+1}^W(1, x_1, \ldots, x_{i-1}, D(x_i), x_{i+1}, \ldots, x_n) =$$
$$(1 - \sum_{i=1}^n x_i \partial_i) D(1) + \sum_{i=1}^n Jac_n^S(x_1, \ldots, x_{i-1}, D(x_i), x_{i+1}, \ldots, x_n).$$

Thus,

$$\sum_{i=1}^{n} Jac_n^S(x_1,\dots,x_{i-1},D(x_i),x_{i+1},\dots,x_n) = \sum_{i=1}^{n} x_i\partial_i D(1).$$

Therefore, $D(1)$ is a linear combination of elements of the form $Jac_n^S(u_1,\dots,u_n)$ and by lemma 8.3, (10) is true.

So, according to (10) $R = \sum_{\alpha\in\mathbf{Z}^n,\alpha\neq-\theta}\lambda_\alpha x^\alpha$, for some $\lambda_\alpha \in K$. If $\lambda_\alpha \neq 0$, then there exists some $i = i(\alpha)$, such that $\alpha_i \neq -1$. Therefore, by lemma 9.4 there exists a derivation of (U, Jac_{n+1}^W) of the form $G := \sum_{\alpha\in\mathbf{Z}^n}\lambda_\alpha(n(\alpha_i+1)^{-1}x^{\alpha+\epsilon_i}\partial_i - x^\alpha)$. Take $D' = D + G$. It is easy to see that

$$D' \in Der\,(U, Jac_{n+1}^W)$$

and

$$D'(1) = 0.$$

Step 2. Suppose that $D(x^{\bar\alpha}) \neq 0$, for some $\bar\alpha \in \mathbf{Z}_+^n, |\bar\alpha| = t$, but $D(x^\beta = 0$, for all $\beta \in \mathbf{Z}_+^n$, such that $|\beta| < t$. As we have shown above (step 1) we can assume that $t > 0$. Consider separately the cases $t = 1$ and $t > 1$.

Case $t = 1$. By the Leibniz rule for $D \in Der\,(U, Jac_{n+1}^W)$, we have

$$D(Jac_n^S(u_1,\dots,u_n)) - \sum_{l=1}^{n} Jac_n^S(u_1,\dots,u_{l-1},D(u_l),u_{l+1},\dots,u_n) =$$

$$D(Jac_{n+1}^W(1,u_1,\dots,u_n)) - \sum_{l=1}^{n} Jac_{n+1}^W(1,u_1,\dots,u_{l-1},D(u_l),u_{l+1},\dots,u_n) =$$

$$Jac_{n+1}^W(D(1),u_1,\dots,u_n).$$

Thus $D \in Der\,(U, Jac_n^S)$, if $D(1) = 0$. By theorem 8.1, D as a derivation of U under multiplication Jac_n^S can be presented in the form

$$D = \bar D + D_0 + \tilde D,$$

where

$$\bar D = \lambda_{-\theta}D_{-\theta},$$

$$D_0 = \lambda_0\Delta,$$

$$\tilde D = \sum_{i=1}^{n}\lambda_i x^{-\theta+\epsilon_i}\partial_i + \sum_{i<j,\alpha\in\mathbf{Z}^n}\lambda_{i,j}(\alpha)D_{i,j}(x^\alpha),$$

for some $\lambda_0, \lambda_{-\theta}, \lambda_i, \lambda_{i,j}$. Notice that $\tilde D$ is a divergence-free differential operator of first order. By corollary 9.2 any divergence-free derivation D is a derivation of (U, Jac_{n+1}^W). Since $D(1) = 0$, $\lambda_0 = 0$. So, $\bar D = D - \tilde D$ is also a derivation of (U, Jac_{n+1}^W). Prove that $\bar D = 0$. As we mentioned before,

$$Jac_{n+1}^W(x^{-\theta},x_1,\dots,x_n) = (n+1)x^{-\theta}.$$

Therefore,

$$\bar D(x^{-\theta}) = (n+1)^{-1}\bar D(Jac_{n+1}^W(x^{-\theta},x_1,\dots,x_n)). \tag{11}$$

Let $\pi : U \to < 1 >$ be projection to $< 1 >$. Notice that

$$\pi(\bar D(x^{-\theta})) = \lambda_{-\theta},$$

$$\pi(Jac_{n+1}^W(\bar D(x^{-\theta}),x_1,\dots,x_n)) = \lambda_{-\theta},$$

Take projections to $< 1 >$ from the both parts of (11). We find that

$$\lambda_{-\theta} = (n+1)^{-1}\lambda_{-\theta}.$$

So, $n\lambda_{-\theta} = 0$, and $\lambda_{-\theta} = 0$, $\bar{D} = 0$.

Case $t > 1$. Prove that this case is not possible. By lemma 9.5 $\partial_i D(x^{\bar{\alpha}}) = 0$, for any $i = 1, \ldots, n$. So, $D(x^{\bar{\alpha}}) = \lambda \cdot 1$, for some $0 \neq \lambda \in K$. Notice that

$$Jac^W_{n+1}(u, x_1, \ldots, x_n) = (1 - \sum_{i=1}^{n} x_i \partial_i)u,$$

for any $u \in U$. Therefore, by the Leibniz rule,

$$(1 - |\bar{\alpha}|)\lambda = (1 - |\bar{\alpha}|)D(x^{\bar{\alpha}}) =$$
$$D(Jac^W_{n+1}(x^{\bar{\alpha}}, x_1, \ldots, x_n)) =$$
$$Jac^W_{n+1}(D(x^{\bar{\alpha}}), x_1, \ldots, x_n) = \lambda.$$

So, $t\lambda = 0$, and $\lambda = 0$. Contradiction.

Summarize the results obtained in all these steps and cases. We see that any derivation of (U, Jac^W_{n+1}) can be presented as a linear combination of derivations of the form $X - n^{-1}Div\, X$, where $X \in W_n(U)$. □

PROOF OF THEOREM 9.1. . Suppose that J is an ideal of U under $(n+1)$-multiplication $\omega = Jac^W_{n+1}$. Then J is ideal of U under n-multiplication $i(1)\omega = Jac^S_n$. Since (U, Jac^S_n) is simple [**5**], $J = 0$ or $J = U$. So, (U, Jac^W_{n+1}) is simple. Other statements of theorem 9.1 follow from lemma 9.7. □

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INSTITUTE OF MATHEMATICS, ACADEMY OF SCIENCES OF KAZAKHSTAN, ALMATY
S. DEMIREL UNIVERSITY, ALMATY
E-mail address: **askar@math.kz**

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